

Hodge Classes on Abelian Sixfolds

Constructions and a Conditional Proof

Research draft

October 8, 2026

Abstract. We revisit a twenty-one-case research manuscript on complex abelian sixfolds. We give arguments for divisor generation on all powers in the four Type I and two Type II cases, compute the Hodge group and the seven-dimensional middle-degree exceptional contribution in Type III, and analyze the imaginary-quadratic Type IV case. Split Weil algebraicity yields results on all powers, including an explicitly constructed quaternionic family. We give examples, Hodge dimensions, and algebraic generators for all rows. An odd-rank polarization adjustment resolves the mixed Weil plane in the full $5 + 1$ case. A joint Hodge-circle analysis settles the entire $3 + 3$ row. Low-dimensional inputs also settle the $3 + 2 + 1$, $3 + 1 + 1 + 1$, and all surface-elliptic product rows. For quartic and sextic CM endomorphism fields we determine the group alternatives and invariant generators. A real-quadratic polarization adjustment also proves the entire quartic-center row. A split trace polarization, together with a rank bound proved using a two-dimensional rational relation space, settles every simple CM sixfold on all powers. A quaternionic discriminant criterion enlarges the proved Type III strata and gives a family outside every split-subfield construction. Repeated elliptic factors settle a further complete stratum of the $4 + 1 + 1$ row. Explicit dihedral quotients of cyclic cover products algebraize quartic mixed classes in families in both $4 + 2$ and $4 + 1 + 1$, including their special parameters. The residual joint groups and generators in both rows are determined, including CM fourfolds. Hodge-rank-three CM fourfolds are independent of simple CM surfaces; every residual coupling has the same quartic mixed model. Their remaining algebraicity reduces to one nonzero projected cycle. For the residual mixed models, a tensor-contraction argument also shows that this single-cycle condition suffices on all powers. The weighted-cover extension covers seventeen branch orbits and adds two marked rational Hermitian types to the previous examples. Completion of the nonconstant period maps and specialization of their fixed Hodge subvariations cover the represented one-dimensional PEL components, including the known nonsplit sextic-center component. An eighteenfold cyclic-cover construction also proves a nonsplit sextic-center family, with Weil discriminant -2 over $\mathbf{Q}(\sqrt{-3})$. Using the newly released split-eightfold theorem of OpenAI as an external input, we prove a stabilization and divisor-contraction theorem for arbitrary sixfold Weil planes. It completes S7, S8, and S10, including their all-powers assertions. OpenAI's separate CM theorem also settles every CM-fourfold subcase in N2 and N3. The specified complete Kuga-Satake correspondence gives a conditional construction of the mixed cycle for every remaining non-CM N3 model. A determinant stabilization in an auxiliary spin representation, together with the separate degree-four spin-tensor algebraicity assertion and the CM theorem, gives the residual N2 cycle. All twenty-one rows are therefore covered relative to these explicit external inputs. This yields a conditional proof of the rational Hodge conjecture for every complex abelian sixfold. The new external preprints have not been fully independently verified here; neither their proofs nor the arguments proposed here are certified by this manuscript.

Contents

Conventions, scope, and proof infrastructure	3
I The eleven simple rows	14
1 S1: Type I(1) (Proved)	14
2 S2: Type I(2) (Proved)	14
3 S3: Type I(3) (Proved)	15
4 S4: Type I(6) (Proved)	15
5 S5: Type II(1) (Proved)	16
6 S6: Type II(3) (Proved)	17
7 S7: Type III(1) (Proved)	17
8 S8: Type IV(1,1) (Proved)	20
9 S9: Type IV(2,1) (Proved)	21
10 S10: Type IV(3,1) (Proved)	24
11 S11: Type IV(6,1) (Proved)	28
II The ten nonsimple rows	34
12 N1: The $5 + 1$ case (Proved)	34
13 N2: The $4 + 2$ case (Proved)	35
14 N3: The $4 + 1 + 1$ case (Proved)	49
15 N4: The $3 + 3$ case (Proved)	57
16 N5: The $3 + 2 + 1$ case (Proved)	61
17 N6: The $3 + 1 + 1 + 1$ case (Proved)	61
18 N7: The $2 + 2 + 2$ case (Proved)	62
19 N8: The $2 + 2 + 1 + 1$ case (Proved)	62
20 N9: The $2 + 1 + 1 + 1 + 1$ case (Proved)	63
21 N10: The $1 + 1 + 1 + 1 + 1 + 1$ case (Proved)	63
Conditional conclusion and external dependencies	64
References	66

Conventions, scope, and proof infrastructure

All abelian varieties are over $\mathbf{C}$, and all cycle coefficients are rational. For an abelian variety A set

$$B^p(A) = H^{2p}(A, \mathbf{Q}) \cap H^{p,p}(A), \quad D^\bullet(A) = \mathbf{Q}[\mathrm{NS}(A)_{\mathbf{Q}}] \subseteq B^\bullet(A).$$

Our main target is A itself with $\dim A = 6$. Statements about every A^n are optional strengthenings, explicitly distinguished below. Here A^n denotes a product of n copies, so $\dim A^n = 6n$, not 6^n . For product rows, the displayed factors are simple; repeated or isogenous factors are allowed. The twenty-one headings are coarse cases, and an endomorphism or signature condition must not be added silently. The Hodge conjecture in codimension p asserts the surjectivity of $\mathrm{cl} : \mathrm{CH}^p(A)_{\mathbf{Q}} \rightarrow B^p(A)$. We write $P_A(t) = \sum_p \dim_{\mathbf{Q}} B^p(A) t^p$; t counts codimension, not cohomological degree. The condition $B^\bullet(A^n) = D^\bullet(A^n)$ for every n is stronger than the Hodge conjecture for A itself. We retain the original row labels S1–S11 and N1–N10. A result on a specified stratum or family is never counted as a result on its entire coarse row.

Reading the status labels

Proved in a row heading means that a sixfold algebraicity argument covering its whole stated row is supplied, using the cited external results. It does not mean that every joint Hodge group has been classified, or that this draft has passed independent review. There are twenty-one **Proved** rows, relative to the external results specified below. In particular, S7, S8, and S10 use the newly released split-eightfold theorem [14, Theorem 1.1]; the additional CM strata in N2 and N3 use the separate CM theorem [15, Theorem 1.1]. The residual non-CM N3 cases also use the specified full Kuga–Satake algebraicity assertion [17, Theorem 1.1]. The residual non-CM N2 cases use the auxiliary-spin proposition of [19] and the CM theorem; its precise hypotheses and tensor are stated before Theorem 13.19. These are newly released preprints. We use their stated results as external inputs, not as the outcome of a complete independent verification of their proofs. Before these new external inputs were introduced, sixteen whole rows had been established.

Rows	Hodge structure / generators	Algebraicity on the sixfold
S1–S6	Full groups; divisor generators	Whole rows
S7, S8	Groups and exceptional generators	Whole rows via split eightfolds
S10	Two unitary groups; Weil plane or pairings	Whole row via split eightfolds
S9	Two unitary group alternatives	Whole row, including Weil classes
S11	Hodge rank five or six; Weil plane or pairings	Whole row, on all powers
N1, N4–N6	Mixed generators identified	Whole rows
N2	Residual groups and generators determined	Whole row, conditional on auxiliary spin and CM inputs
N3	Residual groups and generators determined	Whole row, conditional on specified KS correspondence
N7–N9	Explicit examples; no uniform joint-group list	Whole rows by external theorem
N10	Arbitrary elliptic products	Whole row

What the twenty-one rows classify

For a simple sixfold let F be the center of $D = \text{End}^0(A)$, and in Type IV write $[F : \mathbf{Q}] = 2e_0$, $[D : F] = d^2$, and $m = \dim_D H_1(A, \mathbf{Q})$. The equality $6 = e_0 d^2 m$ forces $d = 1$, since 6 is squarefree. Thus the Type IV centers have degrees 2, 4, 6, 12. In Type I the real field degree divides 6. In Types II and III, $6 = 2em$, where e is the degree of the real center. Type III with $m = 1$ is excluded for simple complex abelian varieties; see [7, §3.2]. These restrictions give the eleven original simple rows. The ten nontrivial partitions of 6 give the product rows. They classify numerical endomorphism and dimension data, not all Mumford–Tate strata, CM types, signatures, or polarization discriminants.

Isogenies, invariants, and descent

Lemma P.1. Rational Hodge classes, their algebraicity, and the Hodge conjecture on all powers are invariant under isogeny.

Proof. If $f : A \rightarrow A'$ is an isogeny, its pullback is a cohomological ring isomorphism over $\mathbf{Q}$. A quasi-inverse isogeny and the degree formulas show that its inverse preserves the rational span of algebraic cycle classes. Pullback and pushforward are algebraic operations. Apply the same argument to f^n for every n . $\square$

Set $V = H^1(A, \mathbf{Q})$. Then $H^\bullet(A^n, \mathbf{Q}) = \bigwedge^\bullet(V^{\oplus n})$, and

$$B^p(A^n) = \left(\bigwedge^{2p}(V^{\oplus n}) \right)^{\text{Hg}(A)}.$$

Equivalently one can use $\text{MT}(A)$ on the Tate-twisted space $H^{2p}(A^n, \mathbf{Q})(p)$. Untwisted positive-degree cohomology must not be identified with the invariant space of $\text{MT}(A)$: homotheties act on it. The identity $\text{Hg}(A^n) = \text{Hg}(A)$ uses the diagonal action. Furthermore

$$\text{End}_{\text{Hg}(A)} H^1(A, \mathbf{Q}) = \text{End}^0(A)^{\text{op}}.$$

The opposite-algebra convention on cohomology does not affect any commutant or dimension argument below. These facts follow from the equivalence between abelian varieties up to isogeny and polarizable rational weight-one Hodge structures; see [2, §§1–2].

Lemma P.2. Suppose that the complexified Hodge group acts on $V_{\mathbf{C}}$ as a product of full symplectic groups on their standard representations, allowing repeated copies. Then every Hodge class on every power is generated by divisor classes. The same conclusion holds for a product of full general linear groups acting on standard representations and their duals.

Proof. The first fundamental theorem of invariant theory says that tensor invariants for the symplectic group are generated by pairwise alternating contractions; for the general linear group they are generated by evaluations between a standard representation and its dual. The alternating projector from tensor powers to exterior powers is equivariant in characteristic zero. Consequently the exterior invariant algebra is generated by its degree-two invariants: each matching gives the exterior product of its degree-two pairings, up to the signs of the alternating projector. For a product group the construction applies to each independent summand. The rational degree-two invariant space is precisely $\text{NS}(A^n)_{\mathbf{Q}}$, by the Lefschetz (1, 1) theorem. Invariants commute with scalar extension for reductive groups, and the image of each multiplication map on rational invariants also commutes with scalar extension. Surjectivity over $\mathbf{C}$ therefore implies surjectivity over $\mathbf{Q}$. Products of divisor cycles are algebraic. This proves the assertion without assuming that individual split complex pairings are rational. See also [2, Theorem 2.14]. $\square$

The small-dimensional representation argument

We use the weight-one representation theorem: in any irreducible complex constituent of $H^1(A, \mathbb{C})$, every nontrivial simple Lie-algebra factor acts through a classical minuscule representation. A Hodge cocharacter is nontrivial on at most one factor acting in that constituent. All rational simple factors are generated by the rational conjugates of the Hodge cocharacter. These statements are explained and proved in [2, §3, especially (3.9)–(3.13)]; they are not hypotheses invented for this manuscript. The argument also applies to a multiplicity-space constituent after splitting an endomorphism algebra.

Lemma P.3. An irreducible symplectic weight-one constituent of dimension 2, 4, or 6, acted on by a connected semisimple image, has full image Sp_2 , Sp_4 , or Sp_6 , respectively. An irreducible orthogonal constituent of dimension 6 has full image SO_6 . For a symplectic constituent of dimension 12, the only possible complex semisimple Lie-algebra images are $\mathfrak{sp}_{12}$ and $\mathfrak{sl}_2 \oplus \mathfrak{sl}_4$, the latter on $\mathbb{C}^2 \otimes \wedge^2 \mathbb{C}^4$.

Proof. Write an irreducible representation of a semisimple algebra as the tensor product of irreducible representations of its simple ideals. If the tensor product is self-dual, each factor is self-dual. The self-dual classical minuscule representations of dimension at most 12 have the following possibilities, with coincidences between low-rank root systems understood:

dimension	symplectic	orthogonal
2	A_1 standard	none
4	C_2 standard	none
6	C_3 standard	$A_3, \wedge^2 \mathbb{C}^4$
8	C_4 standard	B_3 spin, D_4 vector or half-spin
10	C_5 standard	D_5 vector
12	C_6 standard	D_6 vector

This list follows by inspecting $\wedge^j \mathbb{C}^{\ell+1}$ in Type A, spin representations in Types B and D, and the standard representation in Type C. In Type A self-duality requires $2j = \ell + 1$; its dimension is $\binom{\ell+1}{j}$ and its sign is $(-1)^j$. There is no nontrivial self-dual classical minuscule representation of odd dimension. Thus dimension 6 cannot factor as $2 \cdot 3$, and dimension 4 can only factor as $2 \cdot 2$, whose invariant form is symmetric, not alternating. In dimension 12 the only possible nontrivial factorization is $2 \cdot 6$. An alternating form then requires the six-dimensional factor to be orthogonal, giving the displayed $A_1 \times A_3$ possibility. The listed simple images are the full classical groups on these vector spaces. Connectedness converts Lie-algebra equality into group equality. $\square$

Lemma P.4. Let $G \subseteq \prod_{i=1}^e \mathrm{Sp}(W_i)$ be connected, with $\dim W_i = 2r$ independent of i and with surjective projections. If the W_i are pairwise nonisomorphic G -modules, then G is the full product.

Proof. The semisimple Lie-algebra form of Goursat’s lemma expresses a subdirect product of isomorphic simple algebras as a product of diagonally identified blocks. If a block contains $i \neq j$, their standard representations become isomorphic. Every automorphism of $\mathfrak{sp}_{2r}$ is inner (including $r = 1$), so a twisted diagonal has the same consequence. This contradicts the hypothesis. All blocks therefore consist of one factor. $\square$

Reusable constructions of actual examples

A “very general” example below is specified by a fixed rational polarized endomorphism datum and a nonempty period domain; the generic endomorphism algebra is then justified. It is not an assertion of an explicit arithmetic point or an equation for every family.

Proposition P.5. Let F be totally real of degree e , let $r \geq 1$, and choose $Z_\sigma \in \mathfrak{H}_r$ for every real embedding $\sigma : F \hookrightarrow \mathbf{R}$. In $\mathbf{C}^{er}$ take the lattice which is the image of $\mathcal{O}_F^r \oplus \mathfrak{d}_F^{-1r}$ under

$$(x, y) \mapsto (\sigma(x) + Z_\sigma \sigma(y))_\sigma.$$

The quotient is a polarized abelian variety of dimension er with an F -action. At a very general point its Hodge group is the restriction of scalars of Sp_{2r} and its endomorphism algebra is F ; in particular it is simple.

Proof. The trace alternating form $\mathrm{Tr}_{F/\mathbf{Q}}(x \cdot y' - y \cdot x')$ is integral on the given lattice. The symmetry and positive imaginary part of each Z_σ give the Riemann bilinear relations. Multiplication by $\mathcal{O}_F$ preserves the lattice and the complex structure. The family is the Hilbert–Siegel PEL family attached to this datum. Its connected monodromy is Zariski dense in $\mathrm{Res}_{F/\mathbf{Q}} \mathrm{Sp}_{2r}$: after imposing a level, it is an arithmetic subgroup of this group, which has no compact real factors, so Borel density applies. Connected monodromy is contained in the generic Hodge group. The reverse inclusion follows from the polarization and F -action. Equality follows. The centralizer on the standard restriction-of-scalars module is F . Extra Hodge tensors occur in a countable union of proper Hodge loci; outside it the claimed group and endomorphism algebra hold. A nontrivial isogeny decomposition would give a nontrivial rational idempotent in F , which is impossible. The PEL and Hodge-locus facts used here are reviewed in [2]. $\square$

Proposition P.6. Let $D_0 = (-1, 3)_{\mathbf{Q}}$, with generators $i^2 = -1$, $j^2 = 3$, $ij = -ji$. Let $F = \mathbf{Q}$ or $\mathbf{Q}(2 \cos(2\pi/7))$, and set $D_F = D_0 \otimes_{\mathbf{Q}} F$. For $n = 3/[F : \mathbf{Q}]$, on D_F^n put

$$\psi(x, y) = \mathrm{Tr}_{F/\mathbf{Q}} \mathrm{Trd}_{D_F/F} \left(\sum_k \bar{x}_k i y_k \right).$$

Choose an order lattice and multiply ψ by a positive integer to make it integral. The compatible complex structures commuting with left D_F form a nonempty PEL period domain $\mathfrak{H}_n^{[F:\mathbf{Q}]}$. Very general members are simple sixfolds of Type II with endomorphism algebra D_F .

Proof. The quaternion algebra D_0 splits at infinity and is a division algebra: its Hilbert symbol at 3 is $(-1, 3)_3 = -1$, since -1 is a nonsquare modulo 3. An odd-degree field extension cannot split a quaternion division algebra, so D_F remains division. It splits at every real place of F . Quaternion conjugation shows that ψ is alternating and nondegenerate. The adjoint on the left D_F -action is the positive involution $a \mapsto i^{-1} \bar{a} i$. A base compatible complex structure is right multiplication by $-i$. To check positivity, at a real place identify i with $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in $M_2(\mathbf{R})$. Then the contribution to $\psi(x, x(-i))$ is $\mathrm{tr}(x^T x) > 0$ for $x \neq 0$. The Riemann relations therefore give abelian varieties. The centralizer of the datum is the symplectic group on the Morita multiplicity space, and the nonempty domain is the stated product of Siegel domains. Arithmetic monodromy, Borel density, and the commutant argument used in Proposition P.5 give the generic Hodge group and endomorphism algebra. Division of D_F implies simplicity. $\square$

Proposition P.7. Fix a partition $6 = d_1 + \cdots + d_k$. In the product of Siegel moduli spaces $\prod_i \mathcal{A}_{d_i}(N)$ choose a very general tuple $(B_1, \dots, B_k)$, with $N \geq 3$, and put $A = \prod_i B_i$. Each B_i is simple, $\mathrm{End}^0(B_i) = \mathbf{Q}$, the factors are pairwise nonisogenous, and

$$\mathrm{Hg}(A) = \prod_i \mathrm{Sp}_{2d_i}, \quad \mathrm{End}^0(A) = \mathbf{Q}^k, \quad P_A(t) = \prod_i (1 + t + \cdots + t^{d_i}).$$

All Hodge classes on all powers of A are divisor-generated.

Proof. The product universal family has independent arithmetic monodromy in each symplectic factor. Borel density gives the full product as connected monodromy; the polarization decompositions give the reverse containment for the generic Hodge group. At a very general point these groups agree. The commutant is $\mathbf{Q}^k$, so each factor is simple and there are no cross-factor homomorphisms. Lemma P.2 proves the all-powers statement. On one copy, the Sp_{2d_i} invariants of the exterior algebra are $1, \theta_i, \dots, \theta_i^{d_i}$; independence of the factors proves the polynomial formula. On powers the additional degree-two generators are Poincare divisor classes between repeated copies of the same B_i . $\square$

Promoting algebraic Weil classes to all powers

We retain the following established geometric input:

Theorem P.8 (Markman). For a polarized abelian sixfold of Weil type over an imaginary quadratic field, the Weil classes are algebraic if its Hermitian discriminant is -1 in $\mathbf{Q}^\times / N_{K/\mathbf{Q}} K^\times$.

This is [11, Theorem 1.5.1], also stated as the split sixfold case of [13, Theorem 1.2]. It is used as an external theorem, not reproduced as a new proof here. Its discriminant hypothesis is essential. For the standard hyperbolic Hermitian form of rank 6, that discriminant is $(-1)^3 = -1$.

Theorem P.9 (New external inputs). We additionally use the following two statements, with the preprint status specified above.

1. If X is an abelian eightfold with an imaginary quadratic k -action, compatible polarization, and hyperbolic rational k -Hermitian form of signature $(4, 4)$, then its entire rational Weil plane $W_k(X) \subset H^8(X, \mathbf{Q})$ is algebraic. This includes every compatible polarization type and all special periods.
2. Every rational Hodge class on every complex CM abelian variety is algebraic, in every dimension and codimension.

These are respectively [14, Theorem 1.1] and [15, Theorem 1.1]. They are distinct inputs: the first applies to split Weil-type eightfolds that need not be CM. Neither statement is reproved in this manuscript.

Lemma P.10 (Rational hyperbolicity). Let $k/\mathbf{Q}$ be imaginary quadratic. A nondegenerate k -Hermitian form of rank $2r$ and signature (r, r) is hyperbolic if and only if its determinant class is $[(-1)^r]$ in $\mathbf{Q}^\times / N_{k/\mathbf{Q}}(k^\times)$.

Proof. A hyperbolic plane has determinant -1 , proving necessity. For sufficiency, the underlying rational quadratic form $x \mapsto h(x, x)$ is nondegenerate of dimension $4r$ and signature $(2r, 2r)$. When $r \geq 2$, Meyer's theorem for indefinite rational quadratic forms of dimension at least five supplies a nonzero isotropic vector; compare the same reduction in [16, Lemma 3.2]. Choose a second vector pairing to 1 with it and subtract half its self-pairing times the first vector. The resulting two isotropic vectors span a hyperbolic Hermitian plane. Its orthogonal complement has signature $(r-1, r-1)$ and determinant class $[(-1)^{r-1}]$. Iterating reduces to rank two. Write that last form as $\langle a, b \rangle$ with $a, b \in \mathbf{Q}^\times$. The determinant condition gives $-ab = N_{k/\mathbf{Q}}(t)$ for some $t \in k^\times$, and $(t/a, 1)$ is isotropic, since $aN_{k/\mathbf{Q}}(t/a) + b = 0$. It splits the final hyperbolic plane. $\square$

Theorem P.11 (All sixfold Weil planes from the eightfold input). Assume the split-eightfold input in Theorem P.9. Let A be any abelian sixfold of Weil type over an imaginary quadratic field k , with a compatible polarization and signature $(3,3)$. Then every class in $W_k(A) \subset H^6(A, \mathbf{Q})$ is algebraic, without a discriminant restriction.

Proof. Choose a k -basis of $H_1(A, \mathbf{Q})$ and let $d < 0$ be the determinant of the associated Hermitian form h_A . Take k -CM elliptic curves E_+ and E_- with opposite k -types. Their rank-one Hermitian coefficients have opposite signs. Positive rational rescaling of their polarizations therefore gives $P = E_+ \times E_-$ the Hermitian form $\text{diag}(1, d)$. The product $X = A \times P$ has signature $(4,4)$ and determinant $d^2 = N_{k/\mathbf{Q}}(d)$. Lemma P.10 makes its Hermitian form hyperbolic. Clear denominators in the product polarization by a positive integer multiple; hyperbolicity is unchanged. The external eightfold theorem now makes all of $W_k(X)$ algebraic.

It remains to give an algebraic map recovering $W_k(A)$. For the two embeddings of k write

$$W_k(A)_{\mathbf{C}} = L_+ \oplus L_-, \quad W_k(P)_{\mathbf{C}} = M_+ \oplus M_-.$$

Here $L_{\pm} = \bigwedge^6 H^1(A, \mathbf{C})_{\pm}$ and $M_{\pm} = \bigwedge^2 H^1(P, \mathbf{C})_{\pm}$ are one-dimensional. The top exterior power of each embedding summand of $H^1(X) = H^1(A) \oplus H^1(P)$ gives, in the Kunneth decomposition,

$$W_k(X)_{\mathbf{C}} = (L_+ \otimes M_+) \oplus (L_- \otimes M_-).$$

The surface P has signature $(1,1)$, so $W_k(P)$ consists of rational $(1,1)$ -classes. Choose $0 \neq \beta \in W_k(P)$ and a rational divisor D with $\text{cl}(D) = \beta$, using Lefschetz $(1,1)$. Both complex components β_+, β_- are nonzero: they are conjugate, and β is rational and nonzero. The cup-product pairing $M_+ \otimes M_- \rightarrow H^4(P, \mathbf{C})$ is nonzero, since it is the exterior product of the two complementary two-dimensional embedding spaces of $H^1(P, \mathbf{C})$. The same-sign pairings vanish.

Define the rational algebraic correspondence

$$C_{\beta}(\gamma) = (\text{pr}_A)_*(\gamma \cup \text{pr}_P^* \beta).$$

On each summand $L_{\pm} \otimes M_{\pm}$ it is contraction with $\beta_{\mp}$, a nonzero scalar map onto $L_{\pm}$. Consequently $C_{\beta} : W_k(X) \rightarrow W_k(A)$ is an isomorphism over $\mathbf{Q}$, as can be checked after complexification. For $w \in W_k(A)$ choose its rational preimage γ and $Z \in \text{CH}^4(X)_{\mathbf{Q}}$ with $\text{cl}(Z) = \gamma$. Then

$$(\text{pr}_A)_*(Z \cdot \text{pr}_P^* D) \in \text{CH}^3(A)_{\mathbf{Q}}$$

has class w . The codimension is $4 + 1 - \dim P = 3$. Thus the argument uses an actual algebraic contraction, not merely an abstract isomorphism of Hodge structures. $\square$

Lemma P.12 (Products of CM factors). Assume the CM input in Theorem P.9. Every finite product of CM abelian varieties satisfies the rational Hodge conjecture on all its powers, without a Hodge-group independence assumption.

Proof. Products and powers of CM abelian varieties are CM: the products of their commutative semisimple endomorphism algebras have dimension twice that of the corresponding product variety. Apply the external CM theorem to each such power, and use Lemma P.1 if needed. $\square$

Lemma P.13. Let A have dimension 6, imaginary quadratic multiplication by K , and $\text{Hg}(A)_{\mathbf{C}} = \text{SL}_6$ on $W \oplus W^*$. If $W_k(A) = \bigwedge_K^6 H^1(A, \mathbf{Q})$ is algebraic, then the Hodge conjecture holds for every power A^n .

Proof. For SL_6 the tensor invariant theorem gives evaluation pairings and the two determinant tensors. After alternation, evaluation pairings give rational divisor classes, while determinant invariants in degree 6 on n copies have multiplicity spaces $\text{Sym}^6 \mathbb{C}^n$ for W and for W^* . This last assertion also follows from the exterior Cauchy formula: the partition (1^6) on W has transpose (6) on the copy space. For $a = (a_1, \dots, a_n) \in K^n$ consider the rational homomorphism $f_a : A^n \rightarrow A$, $\sum_i a_i x_i$. Clear denominators to obtain an actual homomorphism; on degree 6 this changes its pullback by a nonzero rational scalar. The pullbacks of $W_K(A)$ contain the pure powers $(\sigma(a_i))_{i=1}^n \otimes^6$ in each determinant multiplicity space. The image of K^n under its two embeddings is Zariski dense in $\mathbb{C}^n \times \mathbb{C}^n$, because $K \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C} \times \mathbb{C}$ and a rational vector space is Zariski dense in its complexification. Pure powers span $\text{Sym}^6 \mathbb{C}^n$ in characteristic zero. Thus these algebraic pullbacks span all determinant generators. Products with divisor classes span every invariant. The rational descent argument in Lemma P.2 completes the proof. This argument does not infer algebraicity from invariant theory; the algebraicity of $W_K(A)$ is its geometric input. $\square$

Six-dimensional reductions and geometric inputs

Lemma P.14. For a smooth projective sixfold, the rational Hodge conjecture follows from its cases in codimensions two and three.

Proof. Codimension one is Lefschetz $(1,1)$. For an ample rational divisor L , Hard Lefschetz gives $L^2 : H^4 \simeq H^8$ and $L^4 : H^2 \simeq H^{10}$, with their respective Hodge bidegree shifts. The inverse images of rational Hodge classes are rational Hodge classes. If those inverse images are algebraic, multiplication by the indicated powers of L makes the original classes algebraic. Degrees zero and twelve are generated by the fundamental and point classes. No algebraicity of a Lefschetz inverse is needed in this argument. $\square$

Theorem P.15 (Low-dimensional inputs). We use the following established results with rational coefficients.

1. A simple abelian variety of prime dimension has full Lefschetz Hodge group and divisor-generated Hodge rings on every power [3, Theorem (2.7)] (Tankeev–Ribet).
2. If $\text{Hom}(E, X) = 0$ for an elliptic curve E , failure of $\text{Hg}(X \times E) = \text{Hg}(X) \times \text{Hg}(E)$ requires that E have CM by an imaginary quadratic field embedding in a simple field component of the center of $\text{End}^0(X)$ [3, Proposition (3.8)].
3. The Hodge conjecture holds for products of abelian varieties of dimension at most two [6, Proposition 2.18 in the preprint].
4. All Weil classes on abelian fourfolds of Weil type are algebraic, for every imaginary quadratic field and every discriminant [11, proof of Corollary 1.6.1].

These are external inputs, not new theorems of this manuscript. In addition, the Hodge conjecture holds for every abelian fourfold [11, Corollary 1.6.1]. The second assertion will only be applied after excluding elliptic isogeny factors; the third includes repeated and CM factors.

Lemma P.16 (Odd-rank polarization adjustment). Suppose an imaginary-quadratic Weil sixfold is K -linearly isogenous to $X \times Y$, where the K -rank r of $H_1(X, \mathbb{Q})$ is odd. Then its Weil classes are algebraic.

Proof. Pull back a K -compatible polarization of the given Weil sixfold along the K -linear isogeny. Its restrictions to X and Y are ample and K -compatible. Take the product of these

restricted polarizations. Its total K -Hermitian form has signature $(3, 3)$ and determinant $d < 0$ in a K -basis. Scale the polarization on X by the positive rational number $c = -d$, keeping the other factor fixed. The new determinant is

$$d' = c^r d = -d^{r+1} = -N_{K/\mathbf{Q}}(d^{(r+1)/2}).$$

Thus its discriminant is -1 . Clear denominators in the polarization; a common integral scaling changes the determinant by a sixth power, which is a norm. The Weil space depends on (A, K) and not on this choice of polarization. Theorem P.8 and isogeny invariance finish the proof. $\square$

Lemma P.17 (Real-quadratic polarization adjustment). Let a sixfold A be of Weil type for an imaginary quadratic field k . Suppose its endomorphisms contain $K = kF$, where $F = \mathbf{Q}(\sqrt{d})$ is real quadratic, $d > 0$, and a polarization has Rosati involution equal to complex conjugation on K . Then A admits a k -compatible polarization of discriminant -1 . Consequently its k -Weil classes are algebraic.

Proof. The K -rank of $H_1(A, \mathbf{Q})$ is three. Fix a purely imaginary $\delta \in k^\times$. Express the polarization as $\psi = \text{Tr}_{K/\mathbf{Q}}(\delta H)$ for a nondegenerate K/F -Hermitian form H on K^3 ; the equivalent k -Hermitian form is $h = \text{Tr}_{K/k} H$. Such H is obtained by the nondegenerate trace pairing and the Rosati identity. Put $b = \det_K H \in F^\times$. In the F -basis $(1, \sqrt{d})$ the trace Gram matrix is $\text{diag}(2, 2d)$, of determinant $4d$. Taking the trace of a rank-three Hermitian form therefore gives

$$D := \det_k h = (4d)^3 N_{F/\mathbf{Q}}(b).$$

For completeness, diagonalize H over K with diagonal entries in F and apply the rank-one trace identity to each entry. A change of K -basis changes both sides by the same norm factor, so this proves the formula in the induced k -basis.

The Weil signature of h is $(3, 3)$, hence $D < 0$ and the two real conjugates of b have opposite signs. The element $a = \sqrt{d}b$ or its negative is therefore totally positive in F . Replace the polarization by $\psi_a(x, y) = \psi(ax, y)$. The element a is Rosati-self-adjoint, and its two positive real eigenvalues show that ψ_a remains positive. It is rational; an integral multiple is a polarization. Its Hermitian form is aH , and $N_{F/\mathbf{Q}}(a) = -dN_{F/\mathbf{Q}}(b)$. Thus its determinant over k is

$$\begin{aligned} D' &= (4d)^3 N(b) N(a)^3 \\ &= -64d^6 N(b)^4 = -(8d^3 N(b)^2)^2. \end{aligned}$$

This represents -1 modulo $N_{k/\mathbf{Q}}(k^\times)$. Clearing denominators multiplies it by a sixth power, also a norm. The Weil space is independent of the polarization, so Theorem P.8 proves its algebraicity. $\square$

Lemma P.18 (A split trace polarization for CM Weil sixfolds). Let A be a simple CM sixfold with $K = \text{End}^0(A)$, and let $k \subset K$ be imaginary quadratic with signature $(3, 3)$ on A . Then A admits a compatible split k -Weil polarization. Consequently $W_k(A)$ consists of algebraic classes.

Proof. Put $F = K^+$, so $[F : \mathbf{Q}] = 6$, $K = kF$, and identify $H_1(A, \mathbf{Q})$ with K . A polarization has Rosati involution equal to conjugation on K . Fix $0 \neq \delta \in k$ with $\bar{\delta} = -\delta$. The polarization pairing can therefore be written

$$\psi(x, y) = \text{Tr}_{K/\mathbf{Q}}(\delta b x \bar{y}), \quad b \in F^\times.$$

Indeed the trace pairing identifies a K -compatible alternating form with an anti-invariant element of K , and every such element is δb . The associated k -Hermitian form is $h(x, y) =$

$\text{Tr}_{K/k}(bx\bar{y})$. Over a fixed complex embedding of k it diagonalizes with entries $\sigma(b)$ for the six embeddings $\sigma : F \hookrightarrow \mathbf{R}$. Its signature $(3,3)$ means that exactly three of these entries have each sign.

Order the six embeddings so that their prescribed signs alternate, starting with a negative sign. By weak approximation choose $\theta \in F$ with $\sigma_1(\theta) < \dots < \sigma_6(\theta)$ in this chosen order. One elementary justification is that an embedding matrix of a rational basis identifies F with an invertible real-linear image of $\mathbf{Q}^6$, which is dense in $\mathbf{R}^6$; the strict ordering specifies a nonempty open set. These six values are distinct, so θ generates F . Let f be its monic minimal polynomial and put $b' = 1/f'(\theta)$. The derivative at the j th ordered root has sign $(-1)^{6-j}$. Thus $a = b'/b$ is totally positive. The form $\psi'(x, y) = \psi(ax, y)$ remains a rational polarization: a is Rosati-self-adjoint and positive on each complex eigenspace. An integral positive multiple is an actual polarization on A .

Lagrange interpolation, followed by comparison of the coefficient of T^5 , gives the exact identities

$$\text{Tr}_{F/\mathbf{Q}}\left(\frac{\theta^m}{f'(\theta)}\right) = 0 \quad (0 \leq m \leq 4), \quad \text{Tr}_{F/\mathbf{Q}}\left(\frac{\theta^5}{f'(\theta)}\right) = 1.$$

For $h'(x, y) = \text{Tr}_{K/k}(b'x\bar{y})$ it follows that $L = k\langle 1, \theta, \theta^2 \rangle$ is totally isotropic: each pairing of its indicated generators is one of the first five traces. The generators are independent over k , since $[K : k] = 6$. The trace form is nondegenerate, so L is a maximal isotropic k -subspace. This proves splitting directly, without a local-global argument for norms. Clearing denominators preserves L and splitting. The space $W_k(A)$ does not depend on the polarization, and Theorem P.8 proves its algebraicity. $\square$

Lemma P.19 (A prime-dimensional factor and an elliptic curve). Let B be simple of dimension $g \in \{3, 5\}$ and let E be elliptic. All Hodge classes on $B \times E$ are divisor-generated except possibly for a single rational Weil plane in codimension $(g+1)/2$. The exceptional case occurs precisely when E has CM by a field $k \subset \text{End}^0(B)$ whose signature on B is $((g+1)/2, (g-1)/2)$, up to interchange. In that case

$$B^\bullet(B \times E) = D^\bullet(B \times E) + W_k(B \times E),$$

where the k -action on E is chosen to balance the signature. Every class in the displayed space is algebraic.

Proof. By the prime-dimensional theorem, the possible endomorphism algebras are $\mathbf{Q}$, a totally real field of degree g , an imaginary quadratic field, or a CM field of degree $2g$; the Hodge group is the full Lefschetz group. Indeed the dimension formulas exclude Types II and III and force $d = 1$ in Type IV. If the joint group is a product, its invariants are tensor products of the divisor-generated factor invariants. By Theorem P.15(2), only CM elliptic E with field $k \subset \text{End}^0(B)$ remains. There is at most one such k , since 4 does not divide $2g$.

Write $G = \text{Hg}(B)$, and let $\chi : G \rightarrow U_k$ be the k -determinant on $H^1(B, \mathbf{Q})$. In the CM case $G = U_F$ and $\chi = N_{F/k}$. Orient the k -embedding on E so its Hodge-circle character is z . If k acts on B with signature $(r, g-r)$, the determinant character of its Hodge circle is z^q , with $q = 2r - g$ (a simultaneous reversal of signs is harmless). Simplicity excludes a pure signature: in that case all of $\text{End}_k H_1(B, \mathbf{Q})$ would commute with its complex structure, forcing an elliptic isogeny decomposition. Thus $0 < |q| \leq g-2$. The joint Hodge group is

$$H = \{(v, u) \in G \times U_k : \chi(v) = u^q\}.$$

To justify equality, the displayed relation holds on the Hodge circle, its kernel is connected, and it projects surjectively onto G . Connectedness follows over $\mathbf{C}$ from the primitive character

vector: $\det$ on GL_g , or $(1, \dots, 1)$ on the split CM torus. The actual Hodge group also projects onto G , hence has dimension at least $\dim G = \dim H$ and must equal H .

For quadratic $\mathrm{End}^0(B)$, $\ker \chi$ is SL_g after complexification. Its invariants on $\wedge(W \oplus W^*)$, $\dim W = g$, are pairings and the two volumes. Pairings have degree two and trivial χ -weight; the volumes have degree g and weights $\chi^{\pm 1}$. For CM F , use embedding coordinates $x_1, \dots, x_g$ selected over k . A monomial invariant under $\ker N_{F/k}$ has exponent differences $m(1, \dots, 1)$. On one copy each difference is in $\{-1, 0, 1\}$, so either $m = 0$ and the monomial consists of conjugate pairs, or $m = \pm 1$ and it is a volume of degree g .

On E , only $H^1(E)$ has nonzero weights, namely $u^{\pm 1}$. A volume from B can therefore be neutralized exactly when $|q| = 1$. The two resulting complex lines give the rational k -Weil plane of $B \times E$. No further factor can multiply a volume nontrivially, since it already uses every selected B coordinate and one E coordinate. Rational descent gives the stated decomposition. For $g = 3$ the plane is algebraic by the fourfold theorem; for $g = 5$ apply Lemma P.16 to the rank-one elliptic summand. $\square$

Lemma P.20 (A threefold and repeated elliptic factors). Let B be a simple threefold, and P a product of elliptic curves. Then $B \times P$ satisfies the Hodge conjecture. Its Hodge ring is generated by divisors and pullbacks of the fourfold Weil plane on $B \times E$, if there is an elliptic isogeny class E in P with $k = \mathrm{End}^0(E) \subset \mathrm{End}^0(B)$ imaginary quadratic.

Proof. Collect elliptic isogeny classes. All classes not matching such a k split off in the Hodge group by successive applications of Theorem P.15(2); for a new elliptic curve its CM field would have to embed in a simple component of the center of the existing endomorphism algebra. Distinct elliptic classes have distinct CM fields, and the threefold contains at most one imaginary quadratic field. Non-CM elliptic classes split off by the same proposition. If there is no matching field, factorwise divisor generation proves the claim. Otherwise it suffices to treat $B \times E^m$.

The joint group is the group from Lemma P.19 with $g = 3$ and $|q| = 1$, acting identically on the m copies of E . The kernel-of-determinant invariants on the single B factor consist of pairings and the two degree-three volumes, by that lemma's proof. For a volume, central neutrality requires elliptic weight ∓ 1 . An elliptic exterior monomial of that weight is a product of one unpaired elliptic coordinate and weight-zero pairings. Therefore all invariants are generated by divisors and a volume times one elliptic coordinate. Pullbacks of $W_k(B \times E)$ under $(b, e_1, \dots, e_m) \mapsto (b, \sum a_i e_i)$, $a_i \in k$, span these last generators: after scalar extension the two embeddings of k^m are independent and Zariski dense. Clear denominators in the maps. The fourfold theorem makes all these pullbacks algebraic, and rational descent finishes the proof. Only one copy of B was used. $\square$

Lemma P.21 (A surface separates from a threefold and elliptic curves). If S is a simple abelian surface, B a simple threefold, and P a product of elliptic curves, then $\mathrm{Hg}(S \times B \times P) = \mathrm{Hg}(S) \times \mathrm{Hg}(B \times P)$.

Proof. If S is not CM, apply [3, Proposition (2.4) and Lemma (3.4)]: its Hodge Lie algebra is simple over $\mathbf{Q}$, of noncompact type, with unique irreducible length-one representation. There is no homomorphism from S to $B \times P$, since their simple factors have different dimensions.

If S is CM, its Hodge group is the simple rank-two torus U_F for a primitive quartic CM field F . Its splitting-field Galois group is C_4 or D_4 (the latter of order eight), by [3, §4]. A failure of the asserted product equality would, by Goursat, make U_F an isogeny quotient of the center of $\mathrm{Hg}(B \times P)$. Adding elliptic factors only adds rank-one quadratic Galois constituents, so it suffices to examine the center for B . For real multiplication this center is finite, and for quadratic imaginary multiplication its connected part has rank one. For CM B the torus U_L

has rank three and, by [3, Proposition (2.4)], is $\mathbf{Q}$ -simple unless the sextic field L contains an imaginary quadratic k . A simple rank-three torus has no rank-two quotient. In the remaining case the normal closure of L has Galois group $C_2 \times C_3$ or $C_2 \times S_3$, since $L = kL^+$. Neither group has a quotient isomorphic to C_4 or D_4 . Thus U_F cannot be an isogeny constituent of U_L either. Here we use the equivalence, up to isogeny, between rational tori and rational finite-image Galois modules; these modules are semisimple and have the same splitting-field kernel for isogenous tori. This excludes every possible common quotient. $\square$

Lemma P.22 (Small unitary constituents). Let A be a simple sixfold with $\text{End}^0(A) = K$ a CM field of degree $2e$, where $e = 2$ or 3 , and put $q = 6/e$. Then

$$\text{Hg}(A)_{\mathbf{C}}^{\text{der}} = \text{SL}_q^e.$$

Its determinant image $T \subset U_K$ has rank either e or $e - 1$. In the latter case there is an imaginary quadratic subfield $k \subset K$ with $T = \ker(N_{K/k} : U_K \rightarrow U_k)$, and A is of k -Weil type.

Proof. Put $G = \text{Hg}(A)$. Over $\mathbf{C}$ write $H^1(A) = \bigoplus_{j=1}^e (W_j \oplus W_j^*)$ with $\dim W_j = q$. The commutant is $K \otimes \mathbf{C} = \mathbf{C}^{2e}$, so these $2e$ constituents are irreducible and pairwise inequivalent. The connected center acts by scalars on each, and irreducibility therefore also holds for G^{der} . In dimensions three and two the minuscule list forces the projected Lie algebra to be $\mathfrak{sl}_3$ and $\mathfrak{sl}_2$, respectively. A subdirect product consists of diagonally identified blocks, allowing the duality automorphism for $\mathfrak{sl}_3$.

For $e = 2$, suppose the two projected $\mathfrak{sl}_3$'s are identified. Choose the orientation in each conjugate pair so that W_1, W_2 both carry its standard module. The derived Hodge-circle projection is nontrivial: otherwise the Hodge circle and all its rational conjugates would be central, contrary to the definition of G . On both modules this projection has the same eigenvalues. Adding a scalar central weight must yield exactly the two weight-one Hodge-circle weights $+1, -1$. The required scalar is unique, hence the central weights on W_1, W_2 coincide. This equality is Galois-stable: a Galois automorphism acts on the common A_2 diagram either trivially, preserving both standard constituents (possibly interchanging them), or by duality, replacing both by their duals. Writing their central characters as χ_1, χ_2 , the relation is $\chi_1 \chi_2^{-1} = 1$; each Galois conjugate is the same relation or its inverse. Its kernel is therefore a rational subtorus containing the central projection of the Hodge circle. By the minimality defining the Hodge group it contains every central Hodge-circle conjugate. Consequently the connected central torus generated by the conjugates of the Hodge circle also acts equally on W_1, W_2 . These modules would be isomorphic for G , contradicting the commutant. The two SL_3 factors are therefore independent.

For $e = 3$, the diagonal-block partition of the three A_1 factors is Galois-invariant and the real embeddings are transitive. Since three is prime, either all three factors are independent or they form one diagonal block. Write the signatures as $(r_j, 2 - r_j)$. If all $r_j = 1$, the determinant Hodge circle is trivial; its conjugates then force G to be semisimple. Each W_j is self-dual as an A_1 module, so $W_j \cong W_j^*$ for G , again contradicting the commutant. If all r_j are pure (0 or 2), the Hodge circle is scalar in each block and G is a torus, contradicting its irreducible two-dimensional constituents. Thus a pure place and a balanced place both occur. At the former the derived Hodge-circle projection is zero and at the latter it is nonzero; this excludes one diagonal block. The derived group is therefore SL_2^3 .

The determinant cocharacter has coordinates $2r_j - q$. For $e = 2$, both coordinates are odd and nonzero, so $\text{rk } T \geq 1$. For $e = 3$, after division by two the coordinates are in $\{0, 1, -1\}$, with both zero and nonzero entries. A Galois-stable line in the signed permutation representation cannot contain such a vector: transitivity would force its zero-coordinate set to be both empty and nonempty. Thus $\text{rk } T \geq 2$. Since $\text{rk } U_K = e$, only the asserted ranks remain. If T has codimension one, [3, Lemma (3.7)] identifies it with $\ker N_{K/k}$ for an imaginary quadratic subfield k .

The composition of the Hodge circle with the k -determinant is then trivial; its exponent is the difference of the two k -multiplicities. These multiplicities sum to six, so they are $(3, 3)$. $\square$

Part I

The eleven simple rows

1 S1: Type I(1) (Proved)

Scope and status. Full row: divisor generation on all powers, under exactly $\text{End}^0(A) = \mathbf{Q}$. The argument below strengthens the old very-general claim.

Theorem 1.1. *If A is a simple abelian sixfold and $\text{End}^0(A) = \mathbf{Q}$, then $\text{Hg}(A) = \text{Sp}_{12}$, and $B^\bullet(A^n) = D^\bullet(A^n)$ for every $n \geq 1$. On A itself $P_A(t) = 1 + t + \cdots + t^6$.*

Proof. The commutant of $\text{Hg}(A)$ on $V_{\mathbf{C}}$ is $\mathbf{C}$, so $V_{\mathbf{C}}$ is absolutely irreducible. Its connected center acts by scalars, and scalar matrices in Sp_{12} form a finite group. Thus $\text{Hg}(A)$ is semisimple. Lemma P.3 leaves either the full symplectic algebra or the image $\mathfrak{sl}_2 \oplus \mathfrak{sl}_4$ on $\mathbf{C}^2 \otimes \wedge^2 \mathbf{C}^4$. In the latter case the two nonisomorphic complex simple ideals are each Galois-stable, hence are defined over $\mathbf{Q}$ (as normal factors, up to a finite central isogeny). A nontrivial cocharacter on an irreducible representation of a semisimple factor has at least two distinct weights: otherwise its image is scalar, which is impossible for a nontrivial one-parameter subgroup of a finite center. If the Hodge circle were nontrivial on both factors, the weights on the tensor product would include at least three distinct sums. Weight one permits only two. Consequently the circle is trivial on one rational normal factor. Its smallest rational algebraic containing group would omit that factor, contradicting the defining minimality of $\text{Hg}(A)$. The tensor possibility is excluded. Connectedness now gives $\text{Hg}(A) = \text{Sp}_{12}$. Lemma P.2 proves divisor generation on every power. On one standard symplectic module the only exterior invariants are the successive powers of the polarization, giving P_A . $\square$

Example 1.2. Use Proposition P.5 with $F = \mathbf{Q}$, $r = 6$. Explicitly $A_Z = \mathbf{C}^6 / (Z^6 + ZZ^6)$ for $Z \in \mathfrak{H}_6$ very general. The construction proves simplicity and $\text{End}^0(A_Z) = \mathbf{Q}$. The seven one-dimensional Hodge spaces are generated by $1, \theta, \dots, \theta^6$; on powers add the Poincare divisors between the copies.

2 S2: Type I(2) (Proved)

Scope and status. Full row: divisor generation on all powers.

Theorem 2.1. *Let A be a simple sixfold with $\text{End}^0(A) = F$ totally real of degree 2. Then*

$$\text{Hg}(A) = \text{Res}_{F/\mathbf{Q}} \text{Sp}_6, \quad P_A(t) = (1 + t + \cdots + t^3)^2,$$

and $B^\bullet(A^n) = D^\bullet(A^n)$ for every n .

Proof. The polarization is the trace of a nondegenerate F -alternating form. Thus the Hodge group embeds in the indicated restriction of scalars. Over $\mathbf{C}$, V splits into 2 summands V_σ of

dimension 6. Its commutant is $F \otimes \mathbf{C} = \mathbf{C}^2$; reductivity therefore implies that each V_σ is irreducible and the summands are pairwise nonisomorphic. The center of the image on each summand is scalar and hence finite inside its symplectic group. Lemma P.3 gives surjectivity onto every symplectic factor. Lemma P.4 excludes diagonal identifications, proving equality with the full product. Rational descent determines the stated restriction-of-scalars group. Lemma P.2 proves the all-powers conclusion. The exterior invariants on a single V_σ are $1, \theta_\sigma, \dots, \theta_\sigma^3$, proving the polynomial. $\square$

Example 2.2. Set $F = \mathbf{Q}(\sqrt{5})$ and $r = 3$ in Proposition P.5. This specifies the lattice, period matrices, and polarization and proves that a very general member belongs to this simple row. Choose a rational basis $a_1, \dots, a_2$ of F . The rational $(1, 1)$ classes defined by the forms $\psi(a_j x, y)$ are divisor classes and generate $B^\bullet(A)$. After splitting F , the ring is generated by θ_σ with relations $\theta_\sigma^4 = 0$; its dimension in codimension p is the coefficient of t^p in the polynomial above. Endomorphism-translated Poincare divisors complete the generators on powers.

3 S3: Type I(3) (Proved)

Scope and status. Full row: divisor generation on all powers.

Theorem 3.1. *Let A be a simple sixfold with $\text{End}^0(A) = F$ totally real of degree 3. Then*

$$\text{Hg}(A) = \text{Res}_{F/\mathbf{Q}} \text{Sp}_4, \quad P_A(t) = (1 + t + \dots + t^2)^3,$$

and $B^\bullet(A^n) = D^\bullet(A^n)$ for every n .

Proof. The polarization is the trace of a nondegenerate F -alternating form. Thus the Hodge group embeds in the indicated restriction of scalars. Over $\mathbf{C}$, V splits into 3 summands V_σ of dimension 4. Its commutant is $F \otimes \mathbf{C} = \mathbf{C}^3$; reductivity therefore implies that each V_σ is irreducible and the summands are pairwise nonisomorphic. The center of the image on each summand is scalar and hence finite inside its symplectic group. Lemma P.3 gives surjectivity onto every symplectic factor. Lemma P.4 excludes diagonal identifications, proving equality with the full product. Rational descent determines the stated restriction-of-scalars group. Lemma P.2 proves the all-powers conclusion. The exterior invariants on a single V_σ are $1, \theta_\sigma, \dots, \theta_\sigma^2$, proving the polynomial. $\square$

Example 3.2. Set $F = \mathbf{Q}(2 \cos(2\pi/7))$ and $r = 2$ in Proposition P.5. This specifies the lattice, period matrices, and polarization and proves that a very general member belongs to this simple row. Choose a rational basis $a_1, \dots, a_3$ of F . The rational $(1, 1)$ classes defined by the forms $\psi(a_j x, y)$ are divisor classes and generate $B^\bullet(A)$. After splitting F , the ring is generated by θ_σ with relations $\theta_\sigma^3 = 0$; its dimension in codimension p is the coefficient of t^p in the polynomial above. Endomorphism-translated Poincare divisors complete the generators on powers.

4 S4: Type I(6) (Proved)

Scope and status. Full row: divisor generation on all powers.

Theorem 4.1. *Let A be a simple sixfold with $\text{End}^0(A) = F$ totally real of degree 6. Then*

$$\text{Hg}(A) = \text{Res}_{F/\mathbf{Q}} \text{Sp}_2, \quad P_A(t) = (1 + t + \cdots + t^1)^6,$$

and $B^\bullet(A^n) = D^\bullet(A^n)$ for every n .

Proof. The polarization is the trace of a nondegenerate F -alternating form. Thus the Hodge group embeds in the indicated restriction of scalars. Over $\mathbf{C}$, V splits into 6 summands V_σ of dimension 2. Its commutant is $F \otimes \mathbf{C} = \mathbf{C}^6$; reductivity therefore implies that each V_σ is irreducible and the summands are pairwise nonisomorphic. The center of the image on each summand is scalar and hence finite inside its symplectic group. Lemma P.3 gives surjectivity onto every symplectic factor. Lemma P.4 excludes diagonal identifications, proving equality with the full product. Rational descent determines the stated restriction-of-scalars group. Lemma P.2 proves the all-powers conclusion. The exterior invariants on a single V_σ are $1, \theta_\sigma, \dots, \theta_\sigma^1$, proving the polynomial. $\square$

Example 4.2. Set $F = \mathbf{Q}(\zeta_{13} + \zeta_{13}^{-1})$ and $r = 1$ in Proposition P.5. This specifies the lattice, period matrices, and polarization and proves that a very general member belongs to this simple row. Choose a rational basis $a_1, \dots, a_6$ of F . The rational $(1,1)$ classes defined by the forms $\psi(a; x, y)$ are divisor classes and generate $B^\bullet(A)$. After splitting F , the ring is generated by θ_σ with relations $\theta_\sigma^2 = 0$; its dimension in codimension p is the coefficient of t^p in the polynomial above. Endomorphism-translated Poincare divisors complete the generators on powers.

5 S5: Type II(1) (Proved)

Scope and status. Full row: divisor generation on all powers. The old SO_6 model and the claimed Type II volume obstruction are replaced.

Theorem 5.1. *Let A be a simple sixfold whose endomorphism algebra is an indefinite quaternion division algebra over $\mathbf{Q}$. Then $\text{Hg}(A)_\mathbf{C} = \text{Sp}_6$ on $V_\mathbf{C} = U \otimes W$, $\dim U = 2$, $\dim W = 6$, where the group acts trivially on U . Every Hodge class on every power is divisor-generated, and*

$$P_A(t) = 1 + 3t + 6t^2 + 10t^3 + 6t^4 + 3t^5 + t^6.$$

Proof. For Type II the positive involution on D induces a symmetric form on U . The alternating polarization on $U \otimes W$ therefore induces an alternating form on W , so the connected Lefschetz group has complex form $\text{Sp}(W)$; see [7, §3.2]. The commutant $D \otimes \mathbf{C} = M_2(\mathbf{C})$ implies $\text{End}_{\text{Hg}_\mathbf{C}}(W) = \mathbf{C}$, so W is irreducible. The center of the image on W is finite by Schur's lemma and the symplectic form. Lemma P.3 gives full image Sp_6 . Lemma P.2 applies to the $2n$ copies of W on A^n . For the dimension formula use the exterior Cauchy identity

$$\bigwedge^{2p}(U \otimes W) = \bigoplus_{\lambda \vdash 2p} S_\lambda U \otimes S_{\lambda'} W.$$

A summand has a symplectic invariant precisely when $\lambda = (2a, 2b)$, $3 \geq a \geq b \geq 0$. Its invariant multiplicity is $\dim S_{(2a, 2b)} U = 2a - 2b + 1$, and $p = a + b$. Summing these numbers gives the displayed coefficients. $\square$

Example 5.2. Take $D = (-1, 3)_\mathbf{Q}$, $n = 3$, and the explicit trace polarization of Proposition P.6. Very general compatible complex structures give simple Type II(1) sixfolds. The three-dimensional $\text{NS}(A)_\mathbf{Q}$ corresponds to the self-adjoint subspace of the orthogonal Rosati involution. Its divisor classes generate all Hodge spaces with respective dimensions $(1, 3, 6, 10, 6, 3, 1)$.

6 S6: Type II(3) (Proved)

Scope and status. Full row: divisor generation on all powers. The split group is SL_2^3 , not a torus; a nontrivial torus cannot be a derived group.

Theorem 6.1. *Let A be a simple sixfold of Type II with real center F of degree 3. The complexified Hodge group is SL_2^3 , acting on $\bigoplus_{\sigma} U_{\sigma} \otimes W_{\sigma}$ with $\dim U_{\sigma} = \dim W_{\sigma} = 2$. All Hodge classes on all powers are divisor-generated, and $P_A(t) = (1 + 3t + t^2)^3$.*

Proof. Here $\dim_{\mathbf{Q}} D = 12 = \dim_{\mathbf{Q}} H_1(A)$, so the left D -rank is one. The Type II involution gives the symplectic, rather than orthogonal, form on each W_{σ} . The complex commutant is $D \otimes \mathbf{C} = \prod_{\sigma=1}^3 M_2(\mathbf{C})$. Therefore each W_{σ} is irreducible and different multiplicity-space modules are nonisomorphic. Each projected group is $\mathrm{Sp}_2 = \mathrm{SL}_2$, by Lemma P.3. A diagonal identification would create nonzero cross-summand intertwiners, enlarging that commutant. Lemma P.4 therefore gives the full product. Lemma P.2 proves the conclusion. The calculation in S5 with $\dim W = 2$ instead of 6 gives the single-factor polynomial $1 + 3t + t^2$; take its third power. $\square$

Example 6.2. Set $F = \mathbf{Q}(2 \cos(2\pi/7))$ and $D = (-1, 3)_{\mathbf{Q}} \otimes F$. This is an indefinite quaternion division algebra by the odd-degree argument. Proposition P.6 with $n = 1$ gives a nonempty three-dimensional family and simple very general members. Here $\rho(A) = 9$ and the Hodge-space dimensions are $(1, 9, 30, 45, 30, 9, 1)$. The nine rational divisor generators come from a basis of the Rosati-fixed subspace of D ; Poincare divisors and D endomorphisms provide generators on powers.

7 S7: Type III(1) (Proved)

Scope and status. The whole row is proved on all powers using the new external split-eightfold input and Theorem P.11. Quaternionic endomorphisms generate the seven-dimensional exceptional space from a Weil plane. The older split-subfield criterion and its obstructed example are retained to distinguish the two constructions.

Theorem 7.1. *For a simple Type III(1) sixfold, $V_{\mathbf{C}} = U \otimes W$, $\dim U = 2$, $\dim W = 6$, and $\mathrm{Hg}(A)_{\mathbf{C}} = \mathrm{SO}(W)$. Moreover*

$$P_A(t) = 1 + t + t^2 + 8t^3 + t^4 + t^5 + t^6, \quad \dim_{\mathbf{Q}}(B^3(A)/D^3(A)) = 7.$$

On powers, the invariant algebra is generated by degree-two pairings and six-linear determinant invariants on the multiplicity space.

Proof. For the definite quaternion algebra the Rosati involution is canonical quaternion conjugation. Its split alternating form on U leaves a symmetric form on W in the alternating polarization of $U \otimes W$. Thus the connected Lefschetz group is SO_6 , not Sp_6 . The commutant $M_2(\mathbf{C})$ makes W an irreducible constituent. The center of its orthogonal image is finite. Lemma P.3 forces its semisimple image to be SO_6 . For $\mathrm{O}(W)$ the invariant theorem gives pairings; for $\mathrm{SO}(W)$ it adds the volume tensor $\wedge^6 W$. In the exterior Cauchy formula, orthogonal invariants have $\lambda = (a, a)$ on U , $0 \leq a \leq 6$, with one-dimensional multiplicity $S_{(a,a)} U$. These give one divisor-power invariant in each even degree. The volume contribution first occurs in degree 6: its partition on W is (1^6) and its multiplicity on U is $\mathrm{Sym}^6 U$, of dimension 7. In degree 6 these

classes are independent from the orthogonal invariants. There are no additional contributions below degree 6; the hard Lefschetz isomorphisms between degrees $2p$ and $12 - 2p$ determine the higher degrees. This proves the polynomial. The tensor invariant theorem followed by alternation gives the asserted generators on arbitrary powers. $\square$

Theorem 7.2 (Complete Type III row). *Assuming the split-eightfold input, every simple Type III(1) sixfold satisfies the rational Hodge conjecture on every power.*

Proof. Choose a nonzero pure quaternion in the definite algebra D . Its square is negative rational, so it generates an imaginary quadratic field $k \subset D$. The positive Rosati involution on D is canonical conjugation, hence this action is compatible with the polarization. The action on $H^{1,0}(A)$ extends to $D \otimes_{\mathbf{Q}} \mathbf{C} = M_2(\mathbf{C})$; this six-dimensional module is the sum of three standard modules. The two k -eigenvalues therefore have multiplicities $(3, 3)$. Theorem P.11 makes $W_k(A)$ algebraic. In the notation of Theorem 7.1, its two complex lines have multiplicity vectors u_+^6, u_-^6 in $\text{Sym}^6 U$. For every n , pull these classes back under rational homomorphisms $A^n \rightarrow A$, $(x_i) \mapsto \sum_i d_i x_i$, with $d_i \in D$. After clearing denominators these are algebraic maps; the induced nonzero scalar changes do not affect rational cycle spans. The tuples D^n are Zariski dense in $(M_2(\mathbf{C}))^n$. For a fixed $u_+ \neq 0$, the vectors $(d_i u_+)_i$ are therefore Zariski dense in $U^{\oplus n}$, and their sixth powers span $\text{Sym}^6(U^{\oplus n})$. These are all volume generators on A^n . The other generators in Theorem 7.1 are degree-two pairings, which are divisor classes. Products and rational descent prove the assertion. $\square$

Theorem 7.3 (The earlier split-subfield case). *In the setting of Theorem 7.1, suppose that a subfield $K \subset D$ makes the polarized variety a sixfold of split Weil type. Then all Hodge classes on all powers of A are algebraic.*

Proof. Theorem P.8 makes $W_K(A)$ algebraic. After splitting D , K selects two eigenlines ℓ_+, ℓ_- in U . Its complex Weil classes are the determinant tensors with multiplicity vectors ℓ_+^6 and ℓ_-^6 . On A^n use rational homomorphisms $f_d : A^n \rightarrow A$, $\sum_i d_i x_i$, with $d_i \in D$, clearing denominators. The tuples D^n are Zariski dense in $(M_2(\mathbf{C}))^n$ after complexification. For a fixed nonzero $u \in \ell_+$, the vectors $(d_i u)_i$ therefore range Zariski densely in $U^{\oplus n}$. Their sixth pure powers span $\text{Sym}^6(U^{\oplus n})$. Consequently the pullbacks of the algebraic Weil plane span every volume invariant on A^n , not just the two on A . Pairings are divisor classes. Theorem 7.1 and rational descent now prove the assertion. $\square$

Corollary 7.4 (A single exceptional cycle in Type III). *For a simple Type III(1) sixfold A , the Hodge conjecture holds on all powers if and only if there exists $Z \in \text{CH}^3(A)_{\mathbf{Q}}$ with $\text{cl}(Z) \notin \mathbf{Q}\theta^3$, where θ is a polarization.*

Proof. Theorem 7.1 gives $B^3(A) = \mathbf{Q}\theta^3 \oplus \mathcal{E}(A)$ and $\mathcal{E}(A)_{\mathbf{C}} = \text{Sym}^6 U \otimes \det W$. The exceptional summand is primitive: degree eight has only the one-dimensional divisor contribution, and $D^\times$ -equivariance of multiplication by θ (with its reduced-norm character twist) excludes a map from the seven-dimensional irreducible summand to it. Hence its projection is algebraic on Hodge classes, explicitly

$$\text{cl}(Z) \longmapsto \text{cl}(Z) - \frac{\int_A \text{cl}(Z)\theta^3}{\int_A \theta^6} \theta^3.$$

The action of $D^\times$ on $\mathcal{E}(A)$ is absolutely irreducible: after complexification its multiplicity action is the irreducible Sym^6 representation of GL_2 . Rational quaternion elements are Zariski dense in $M_2(\mathbf{C})$, and their pullbacks are algebraic correspondences after clearing denominators. Thus the orbit of any nonzero algebraic exceptional projection spans all of $\mathcal{E}(A)$. Its complex span contains the pure volume vectors u^6 . The D -linear sum maps of Theorem 7.3

then span $\text{Sym}^6(U^{\oplus n})$ on every power. All pairings are divisors, proving sufficiency; necessity follows from $\dim \mathcal{E}(A) = 7$. This criterion does not construct the required Z . $\square$

Example 7.5. Let $D = (-1, -1)_{\mathbf{Q}}$ with $i^2 = j^2 = -1$, and $V = D^3$. Put

$$\psi(x, y) = \text{Trd} \left(\sum_{k=1}^3 \bar{x}_k y_k i \right).$$

On an order lattice, clear denominators. Right multiplication by $-i$ is a compatible complex structure, since $\psi(x, x(-i)) = \text{Trd}(\sum_k \bar{x}_k x_k) > 0$ for $x \neq 0$. The left D -action has Rosati adjoint $a \mapsto \bar{a}$. The compatible PEL domain is $\text{SO}^*(6)/\text{U}(3)$, of complex dimension 3. Its group is noncompact and $\mathbf{Q}$ -simple; arithmetic monodromy is dense in its connected SO_6 complex form. The generic commutant is D ; very general members are therefore simple Type III(1) sixfolds. Take $K = \mathbf{Q}(i)$ acting on the left. The associated Hermitian form can be written as $h(x, y) = \psi(ix, y) + i\psi(x, y)$. On the K -basis $(1, j)$ of each copy of D , it has matrix $\text{diag}(2, -2)$. Thus on D^3 it is three hyperbolic planes; the span of the three vectors $1 + j$ in separate summands is a three-dimensional totally isotropic K -subspace. Its discriminant is -64 , the class of -1 modulo norms. This computation uses the fixed rational polarization, so it remains valid throughout the compatible family. The signature is $(3, 3)$: left j interchanges the two K eigenspaces and commutes with the complex structure. Theorem 7.3 applies. The eight middle Hodge classes consist of θ^3 and seven volume classes; the latter are spanned by D -translates of the algebraic Weil plane.

Remark 7.6. The following criterion determines exactly when this split-subfield method applies. The subsequent example shows that it does not apply to every Type III sixfold.

Proposition 7.7 (An arithmetic split-subfield criterion). *Let A be a simple Type III(1) sixfold with definite quaternion endomorphism algebra D . Diagonalize its rational polarization on $H_1(A, \mathbf{Q}) = D^3$ as*

$$\psi(x, y) = \text{Trd} \left(\sum_{s=1}^3 \bar{x}_s y_s q_s \right), \quad \bar{q}_s = -q_s, \quad q_s \neq 0, \quad N = \prod_{s=1}^3 \text{Nrd}(q_s) > 0.$$

There exists an imaginary quadratic subfield $k \subset D$ for which this polarization has k -Weil discriminant -1 if and only if $\mathbf{Q}(\sqrt{-N})$ embeds in D . When it does, the Hodge conjecture holds on every power of A .

Proof. The displayed diagonalization is the usual orthogonalization of a nondegenerate quaternionic skew-Hermitian form. Fix any imaginary quadratic $k = \mathbf{Q}(i) \subset D$, and choose j with

$$i^2 = -a, \quad j^2 = -b, \quad ij = -ji, \quad a, b \in \mathbf{Q}_{>0}.$$

The canonical Rosati involution makes the polarization compatible with k . Left multiplication by j interchanges its two eigenspaces and commutes with the complex structure, so its signature is $(3, 3)$. For one diagonal entry $q = xi + yj + zij$, the associated k -Hermitian form $h(v, w) = \psi(iv, w) + i\psi(v, w)$, on the basis $(1, j)$, is

$$\begin{pmatrix} 2ax & 2abz - 2byi \\ 2abz + 2byi & -2abx \end{pmatrix}.$$

Its determinant is $-4ab \text{Nrd}(q)$, since $\text{Nrd}(q) = ax^2 + by^2 + abz^2$. Taking the three blocks, and using that a and every rational square are norms from k , gives

$$[\det_k h] = [-bN] \quad \text{in } \mathbf{Q}^\times / N_{k/\mathbf{Q}}(k^\times).$$

Consequently the split condition is $bN \in N_{k/\mathbf{Q}}(k^\times)$. The quaternion algebra $D = (k, -b)$ then also has the presentation $(k, -N)$: the ratio N/b is a norm, so rescale its second generator. This produces an element of square $-N$, proving necessity.

Conversely, if $u \in D$ has square $-N$, choose a nonzero pure quaternion i anticommuting with u . Definiteness gives $i^2 = -a < 0$, and $D = (\mathbf{Q}(i), -N)$. In the preceding determinant calculation take $b = N$; then $bN = N^2$ is a norm. The discriminant is -1 . Theorem 7.3 proves the all-powers assertion. $\square$

Example 7.8 (A quaternionic family outside every split subfield). Take $D = (-2, -5)_{\mathbf{Q}}$, with generators i, j , and $(q_1, q_2, q_3) = (i, j, ij)$. Their reduced norms are 2, 5, 10, hence $N = 100$ and $\mathbf{Q}(\sqrt{-N}) = \mathbf{Q}(\sqrt{-1})$. The algebra D is division at 5: the local Hilbert symbol is $(-2, -5)_5 = (\frac{-2}{5}) = -1$. But $\mathbf{Q}(\sqrt{-1})$ splits at 5, so it cannot embed in D . Indeed tensoring such an embedding with $\mathbf{Q}_5$ would embed $\mathbf{Q}_5 \times \mathbf{Q}_5$ in a division algebra. Thus Proposition 7.7 excludes every split imaginary quadratic subfield for this polarization.

This is a nonempty polarized datum. On the s th real quaternionic coordinate take right multiplication by $-q_s / \sqrt{\text{Nrd}(q_s)}$ as complex structure. Its square is -1 and

$$\psi(x, Jx) = 2 \sum_s \sqrt{\text{Nrd}(q_s)} \text{Nrd}(x_s) > 0 \quad (x \neq 0).$$

The Riemann relations and an order lattice give abelian varieties. The same arithmetic-monodromy and commutant argument as in Example 7.5 gives simple very general members with $\text{End}^0 = D$ in a three-dimensional PEL family. For such a member every rational polarization is a positive rational multiple of this one, because the Rosati-fixed subspace of D is $\mathbf{Q}$. A multiplier c replaces N by $c^6 N$, which defines the same quadratic field. Hence changing polarization cannot remove this obstruction. This is an obstruction to the split-subfield method, not a nonalgebraicity assertion. Nevertheless, Theorem 7.2 now proves the Hodge conjecture on all powers of every simple member, using the eightfold input. For example, choose the subfield $k = \mathbf{Q}(\sqrt{-2}) \subset D$ generated by i and stabilize its sixfold Weil structure as in Theorem P.11; no split imaginary quadratic subfield inside D is required.

8 S8: Type IV(1,1) (Proved)

Scope and status. The whole row is proved on all powers. Unbalanced signatures are divisor-generated. For balanced signature, arbitrary rational Hermitian discriminants are covered by Theorem P.11, using the new external split-eightfold input.

Theorem 8.1. Suppose A is a simple sixfold with $\text{End}^0(A) = K$ imaginary quadratic. For the K -action on $H^{1,0}$ write its signature as $(r, 6-r)$, with $1 \leq r \leq 5$. If $r \neq 3$, then $\text{Hg}(A)$ is the full unitary group and every Hodge class on every power is divisor-generated. If $r = 3$, then $\text{Hg}(A)$ is the special unitary group,

$$P_A(t) = 1 + t + t^2 + 3t^3 + t^4 + t^5 + t^6,$$

and the Hodge conjecture on all powers is equivalent to algebraicity of $W_K(A)$. Under the split-eightfold input it holds for every discriminant, on all powers.

Proof. Over $\mathbf{C}$ write $V = W \oplus W^*$ with $\dim W = 6$. The two summands are irreducible and nonisomorphic for the full Hodge group, since its commutant is $K \otimes \mathbf{C} = \mathbf{C}^2$. Its center acts by scalars on each, so the derived representation on W is irreducible. The classical minuscule

list in dimension 6 leaves: the standard representation of SL_6 , the standard symplectic representation of Sp_6 , the orthogonal $\wedge^2 \mathbf{C}^4$ of SL_4 , or $\mathbf{C}^2 \otimes \mathbf{C}^3$ for $\mathrm{SL}_2 \times \mathrm{SL}_3$. The last case has two different absolute root types and hence two rational normal factors. A nontrivial Hodge cocharacter on both would give at least three weights on W ; one inactive rational factor contradicts minimality of the Hodge group, as in S1. It is excluded. In the symplectic or orthogonal case a nontrivial cocharacter with only two weights has opposite weights of equal multiplicity 3. Adding a central weight does not change those multiplicities, so the signature must be $(3,3)$. For that signature the determinant of K -linear H^1 has zero Hodge-circle weight. As K is imaginary quadratic, all its rational conjugates also have weight zero; hence $\mathrm{Hg}(A) \subset \mathrm{SU}_K(h)$. Its connected center is then finite: a scalar on W with determinant one has sixth power one. Thus W and W^* would be isomorphic for $\mathrm{Hg}(A)$ in either of the self-dual alternatives, enlarging the commutant. This excludes both. The derived image is SL_6 . If $r = 3$, its full group is $\mathrm{SU}_K(h)$. If $r \neq 3$, the determinant weight $2r - 6$ is nonzero; it generates the one-dimensional unitary determinant torus. The full group is therefore $\mathrm{U}_K(h)$. General-linear invariant theory proves divisor generation in the latter case. In the former, contractions give polarization powers and the two determinants give the rational Weil plane, proving the polynomial. Necessity of Weil algebraicity is immediate on A , and sufficiency on all powers is Lemma P.13. Theorem P.11 supplies its algebraicity for every discriminant under the new input. For discriminant -1 , Markman's theorem already suffices. The excluded signatures $(0,6)$ and $(6,0)$ would make the Hodge circle K -scalar on W ; its rational containing group is then a torus with a larger commutant, contradicting simplicity with endomorphism algebra K . $\square$

Example 8.2. Take $K = \mathbf{Q}(i)$, $V = K^6$, and $h = \mathrm{diag}(1, 1, 1, -1, -1, -1)$ with alternating form $\mathrm{Tr}_{K/\mathbf{Q}}(ih)$ on an order lattice. Compatible positive complex structures form the nonempty unitary domain of signature $(3,3)$, of complex dimension 9. The generic monodromy is the full special unitary group, whose commutant is K ; very general members are simple with $\mathrm{End}^0 = K$. The Hermitian form is hyperbolic and has discriminant -1 . The middle Hodge space has dimension 3, generated by θ^3 and an algebraic rational basis of $W_K(A)$. Their pullbacks under K -linear sum maps, together with divisors, generate the Hodge classes on every power. For a comparison of boundaries, replacing the last diagonal entry by -3 gives determinant -3 . The positive number 3 is not a norm from $\mathbf{Q}(i)$ (its 3-adic valuation is odd), so the split theorem does not apply directly to that family. Its product with two oppositely oriented $\mathbf{Q}(i)$ -CM elliptic curves with Hermitian coefficients 1, -3 has signature $(4,4)$ and determinant 9, hence is split. Theorem P.11 contracts its algebraic Weil plane to the original sixfold. Lemma P.13 then proves algebraicity on every power, including this formerly unresolved family.

9 S9: Type IV(2,1) (Proved)

Scope and status. The entire row is proved, including the exceptional Weil case. The real-quadratic polarization adjustment makes its Weil plane split; the determinant argument then also proves algebraicity on all powers.

Theorem 9.1. *Let A be simple with CM endomorphism field K of degree 4. Its derived Hodge group has complex form SL_3^2 . There are exactly the following alternatives for its determinant image.*

1. *The image is all of U_K . Then $\mathrm{Hg}(A)_{\mathbf{C}} = \mathrm{GL}_3^2$, all powers are divisor-generated, and $P_A(t) = (1 + t + t^2 + t^3)^2$.*

2. The image is $\ker N_{K/k}$ for an imaginary quadratic subfield $k \subset K$. Then A is of k -Weil type,

$$B^\bullet(A) = D^\bullet(A) + W_k(A), \quad P_A(t) = (1 + t + t^2 + t^3)^2 + 2t^3.$$

The sum adds a two-dimensional space only in codimension three. Every class is algebraic, and the Hodge conjecture holds on all powers.

Proof. Lemma P.22 gives the derived group and determinant torus. The full Hodge group is the preimage of that torus under the determinant map on the full unitary Lefschetz group: it contains the entire derived kernel, and the two connected groups have the same dimension. For the full image, the general-linear invariant theorem gives pairings. For the norm kernel, orient the K embeddings over one embedding of k . The split determinant relation is $z_1 \cdots z_2 = 1$. The invariant computation in Proposition 9.3 gives pairings and the two simultaneous volumes, with the indicated dimensions. That computation does not require algebraicity. For the geometric assertion, write $F = K^+$ and $K = kF$. A polarization has Rosati conjugation on K , so Lemma P.17 supplies a polarization with k -discriminant -1 and makes the Weil plane algebraic. Proposition 9.3 then proves algebraicity on all powers. $\square$

Example 9.2. Let $K = \mathbf{Q}(\zeta_5)$, $F = \mathbf{Q}(\sqrt{5})$, $V = K^3$, and $h = \text{diag}(1, 1, -1)$. Choose a nonzero totally imaginary $\delta \in K$ and the trace alternating form $\text{Tr}_{K/\mathbf{Q}}(\delta h)$, with compatible positive complex structures and an order lattice. The signatures at the two real embeddings of F are $(2, 1)$. This gives a nonempty PEL domain of complex dimension 4. Arithmetic monodromy gives generic derived group $\text{Res}_{F/\mathbf{Q}} \text{SU}(h)$, whose complex form is SL_3^2 . The determinant Hodge cocharacter is nonzero. On the two-dimensional anti-conjugate character space, a generator of the cyclic Galois group $\text{Gal}(K/\mathbf{Q})$ acts by a signed rotation of order 4; this representation is irreducible over $\mathbf{Q}$. The orbit of any nonzero determinant vector therefore spans that entire space. The generic central torus is full, so the generic Hodge group is the full unitary group. Its centralizer is K ; very general members are simple with $\text{End}^0 = K$. Their Hodge dimensions are $(1, 2, 3, 4, 3, 2, 1)$, generated by the two rational Rosati-fixed divisor classes, with the usual Poincare generators on powers. This verifies the group hypothesis rather than assuming it.

Proposition 9.3. Suppose a sixfold has a CM-field action by K of degree $2e$, $q = 6/e$, $e \in \{2, 3, 6\}$, and

$$\text{Hg}(A)_{\mathbf{C}}^{\text{der}} = \prod_{j=1}^e \text{SL}(W_j), \quad \dim W_j = q.$$

Choose one embedding in each conjugate pair. Suppose the determinant image of $\text{Hg}(A)_{\mathbf{C}}$ is the connected torus $\{(z_1, \dots, z_e) : \prod z_j = 1\}$, and these selected embeddings are exactly those restricting to a fixed embedding of an imaginary quadratic subfield $k \subset K$. If $W_k(A)$ consists of algebraic classes, the Hodge conjecture holds for A^n for every n . In particular this holds when the induced k -Weil structure has discriminant -1 . On A ,

$$P_A(t) = (1 + t + \cdots + t^q)^e + 2t^3.$$

Proof. The exterior invariant algebra for the derived group is generated by pairings between W_j and W_j^* and determinant tensors on q copies of either space. This is the first fundamental theorem for SL_q , applied to the tensor algebra and then its exterior quotient. For $q = 1$ the derived group is trivial and the determinant generators are the individual degree-one coordinates, so the assertion is immediate. A determinant and a dual determinant in the same factor have product equal, up to the alternating convention, to a determinant of pairings. After eliminating such pairs, a central invariant determinant monomial must have the same signed determinant exponent in every factor: the kernel of the characters on the displayed torus is generated by $(1, \dots, 1)$. Consequently the invariant algebra is generated by pairings and products of one

determinant from each factor, all positive or all negative. Those products have cohomological degree $eq = 6$.

On one copy the two products constitute precisely the complexification of the rational k -Weil plane $\bigwedge_k^6 H^1(A, \mathbf{Q})$, embedded in $H^6(A, \mathbf{Q})$. They are algebraic by hypothesis; discriminant -1 is a sufficient way to verify this hypothesis by [11, Theorem 1.5.1]. To treat powers, pull back this whole plane by the algebraic sum maps $A^n \rightarrow A$ with coefficients in K . Clearing denominators realizes all coefficients as endomorphisms. At the selected e embeddings the coefficients vary Zariski densely and independently in $(\mathbf{C}^n)^e$, since $K \otimes_{\mathbf{Q}} \mathbf{C}$ is the product of its embedding coordinates. Thus these pullbacks span $\bigotimes_{j=1}^e \text{Sym}^q(\mathbf{C}^n)$, the space of all positive determinant products; pure q th powers span each symmetric power. The conjugate plane gives all negative products. The pairings are divisor classes, by Lefschetz $(1, 1)$. Surjectivity descends from $\mathbf{C}$ to $\mathbf{Q}$ by scalar extension of the rational cycle-class spans, proving the assertion. On one copy pairings contribute the displayed product polynomial. Only the two simultaneous volumes survive in addition, both in codimension three; this proves the dimension formula. $\square$

Example 9.4 (A split coupled quartic-CM family). Take $K = \mathbf{Q}(i, \sqrt{2})$, $F = \mathbf{Q}(\sqrt{2})$, $k = \mathbf{Q}(\sqrt{-2})$, $V = K^3$, and $h_K = \text{diag}(1, 1, -1)$. Use the alternating form $\psi = \text{Tr}_{K/\mathbf{Q}}(ih_K)$ on an order lattice. The signatures relative to $\mathbf{Q}(i)$ are $(2, 1), (2, 1)$; the compatible PEL domain has dimension four. Its generic derived group is $\text{Res}_{F/\mathbf{Q}} \text{SU}(h_K)$, with complex form SL_3^2 , by arithmetic monodromy. The determinant cocharacter relative to these two embeddings is proportional to $(1, 1)$. The biquadratic Galois group interchanges its coordinates and negates them simultaneously, so its orbit spans exactly this line. This determines the generic central determinant torus.

Relative instead to the two embeddings restricting to one embedding of k , one $\mathbf{Q}(i)$ -embedding is conjugated. Thus the signatures are $(2, 1), (1, 2)$, the central relation is $z_1 z_2 = 1$, and the total k signature is $(3, 3)$. Writing $\delta' = i\sqrt{2}$, the k -Hermitian form associated with the same ψ is

$$h_k = \text{Tr}_{K/k}((1/\sqrt{2})h_K).$$

On the k -basis $(1, \sqrt{2})$ its first two blocks are $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$, and the third is the negative of this block. The span of the first vector in each block is a three-dimensional isotropic k -subspace. This proves splitness explicitly. The generic centralizer on V is K : the two derived standard modules are inequivalent and each is distinguished from its dual by the nontrivial center. Hence a very general member is simple with $\text{End}^0(A) = K$. Proposition 9.3 applies: every power satisfies the Hodge conjecture, while $P_A(t) = 1 + 2t + 3t^2 + 6t^3 + 3t^4 + 2t^5 + t^6$. In particular its two exceptional middle classes are genuinely present and algebraic; this example is not divisor-generated.

Example 9.5 (A nonsplit polarization that can be corrected). Let $k = \mathbf{Q}(i)$, $F = \mathbf{Q}(\sqrt{3})$, $K = kF$, and $H = \text{diag}(1, -1, 1 + \sqrt{3})$ on K^3 . Use $\psi = \text{Tr}_{K/\mathbf{Q}}(iH)$ and a compatible very general complex structure. The signatures are $(2, 1)$ and $(1, 2)$. The product derived monodromy and the determinant-circle relation give the group in the second alternative of S9; its commutant is K and the sixfold is simple. Here $b = -1 - \sqrt{3}$, $N_{F/\mathbf{Q}}(b) = -2$, and the initial k -determinant is

$$D = 12^3(-2) = -3456 = -6 \cdot 24^2.$$

It is not the class of -1 : the factor 3 occurs to odd order and is inert in $\mathbf{Q}(i)$. But $a = 3 + \sqrt{3}$ is totally positive and has norm 6. Replacing H by aH gives

$$D' = D \cdot 6^3 = -746496 = -864^2.$$

This is a polarization of discriminant -1 on the same complex abelian variety. Its Weil plane is unchanged and is therefore algebraic. The Hodge dimensions are $(1, 2, 3, 6, 3, 2, 1)$. This example tests the nontrivial step of Lemma P.17: scaling by a totally positive real-quadratic endomorphism can change the norm class even though a rational scalar on a sixfold cannot.

10 S10: Type IV(3,1) (Proved)

Scope and status. The whole row is proved on all powers using the new external split-eightfold input for its exceptional Weil plane. The full-unitary case is divisor-generated. The explicit split and nonsplit cyclic constructions below remain independent geometric examples of the row.

Theorem 10.1. *Let A be simple with CM endomorphism field K of degree 6. Its derived Hodge group has complex form SL_2^3 . There are exactly the following alternatives for its determinant image.*

1. *The image is all of U_K . Then $\mathrm{Hg}(A)_{\mathbb{C}} = \mathrm{GL}_2^3$, all powers are divisor-generated, and $P_A(t) = (1 + t + t^2)^3$.*
2. *The image is $\ker N_{K/k}$ for an imaginary quadratic subfield $k \subset K$. Then A is of k -Weil type,*

$$B^\bullet(A) = D^\bullet(A) + W_k(A), \quad P_A(t) = (1 + t + t^2)^3 + 2t^3.$$

The sum adds a two-dimensional space only in codimension three. Under the split-eightfold input, every class on every power is algebraic, for arbitrary induced k -Hermitian discriminant.

Proof. Lemma P.22 gives the derived group and determinant torus. The full Hodge group is the preimage of that torus under the determinant map on the full unitary Lefschetz group: it contains the entire derived kernel, and the two connected groups have the same dimension. For the full image, the general-linear invariant theorem gives pairings. For the norm kernel, orient the K embeddings over one embedding of k . The split determinant relation is $z_1 \cdots z_3 = 1$. The invariant computation in Proposition 9.3 gives pairings and the two simultaneous volumes, with the indicated dimensions. That computation does not require algebraicity. Theorem P.11 now makes the entire k -Weil plane algebraic for arbitrary discriminant. Proposition 9.3 then proves algebraicity on all powers. For discriminant -1 , the earlier Theorem P.8 suffices. $\square$

Example 10.2. Set $K = \mathbb{Q}(\zeta_7)$, $F = \mathbb{Q}(\alpha)$ with $\alpha = 2 \cos(2\pi/7)$, and $V = K^2$. Choose $h = \mathrm{diag}(1, \alpha - 1)$ and an integral multiple of $\mathrm{Tr}_{K/\mathbb{Q}}(\delta h)$ for a nonzero totally imaginary δ . The conjugates of α are approximately $1.24698, -0.44504, -1.80194$. Thus the signatures are $(2, 0), (1, 1), (1, 1)$, and the nonempty compatible period domain has complex dimension 2. The rational derived unitary group is simple and has noncompact real factors. Arithmetic monodromy is dense in it, including its compact real factor, so its generic complex form is SL_2^3 . The determinant cocharacter has just one nonzero component in the three-dimensional anti-conjugate character space. The cyclic Galois group of K permutes its three coordinate directions with signs; that vector's orbit spans all three directions. The central determinant torus is full, so the generic Hodge group is GL_2^3 after complexification. The commutant is K , proving simplicity and $\mathrm{End}^0 = K$ for very general members. Their Hodge dimensions are $(1, 3, 6, 7, 6, 3, 1)$; a basis of F gives three rational divisor generators. Although the label is Type IV, these members are not CM points: their Hodge group has nontrivial derived part. Calling the entire row toric would be incorrect.

Example 10.3 (A split coupled sextic-CM family). Let $K = \mathbf{Q}(\zeta_7) = kF$, where $k = \mathbf{Q}(\sqrt{-7})$ and $F = \mathbf{Q}(\alpha)$, $\alpha = 2\cos(2\pi/7)$, $\alpha^3 + \alpha^2 - 2\alpha - 1 = 0$. Set

$$a = \alpha^2 + \alpha - 1, \quad b = \alpha - 1, \quad V = K^2, \quad h_K = \text{diag}(a, b),$$

and use $\text{Tr}_{K/\mathbf{Q}}(\sqrt{-7}h_K)$ on an order lattice. At the three real embeddings the signs of a are $(+, -, +)$ and those of b are $(+, -, -)$, so the signatures are $(2, 0), (0, 2), (1, 1)$. The compatible domain has dimension one and generic derived group $\text{Res}_{F/\mathbf{Q}} \text{SU}(h_K)$, of complex form SL_2^3 . The determinant cocharacter is proportional to $(1, -1, 0)$. The cyclic Galois action on F generates the plane of coordinate sum zero; conjugation negates it. Thus the determinant image is exactly $z_1 z_2 z_3 = 1$. The three standard derived modules are distinct and their central characters are nonzero, so none is identified with its dual; the generic centralizer is K . Very general members are simple with $\text{End}^0(A) = K$.

The induced k -Hermitian form is $h_k = \text{Tr}_{K/k}(h_K)$ and has signature $(3, 3)$. In the basis $(1, \alpha, \alpha^2)$ the trace-form determinant of F is 49. The multiplication matrices are

$$M_a = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad M_b = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 2 \\ 0 & 1 & -2 \end{pmatrix}.$$

Their determinants are -1 and 1 , respectively. Therefore

$$\det(h_k) = 49^2 \text{Norm}_{F/\mathbf{Q}}(ab) = -49^2.$$

In $\mathbf{Q}^* / \text{Norm}_{k/\mathbf{Q}}(k^*)$ this is -1 , since 49^2 is the norm of the rational number 49. This is exactly the discriminant hypothesis of [11, Theorem 1.5.1]; no additional identification of Hermitian forms is needed. Proposition 9.3 gives the Hodge conjecture on all powers and

$$P_A(t) = 1 + 3t + 6t^2 + 9t^3 + 6t^4 + 3t^5 + t^6.$$

The two additional degree-six classes here are the algebraic k -Weil plane. The full-unitary example above has middle dimension seven, so endomorphism degree alone demonstrably does not determine Hodge dimensions.

Lemma 10.4 (All powers of the Fermat-eighteen Jacobian). Let F_{18} be the Fermat curve of degree eighteen. The Hodge conjecture holds for every power of $J(F_{18})$.

Proof. The $(\mu_{18})^2$ characters on $H^1(F_{18})$ have multiplicity one and labels (a, b) with $a, b, a + b \not\equiv 0$ modulo eighteen. Their rational Galois orbits have size two or six. The corresponding abelian factors have CM by subfields of $K = \mathbf{Q}(\zeta_{18}) = \mathbf{Q}(\zeta_9)$ and dimension one or three. The cyclic sextic field K has a unique proper CM subfield $k = \mathbf{Q}(\sqrt{-3})$. Of its eight CM types, two are induced from k , while the other six form one orbit under $\text{Gal}(K/\mathbf{Q})$. Thus there is only one primitive threefold isogeny class X ; an imprimitive threefold factor is isogenous to E^3 , where E has CM by k . Consequently $J(F_{18}) \sim X^a \times E^b$. Corollary 1.11 of [9] proves the Hodge conjecture for every $X^r \times E^s$, and hence the assertion. The exact character enumeration gives $a = 36, b = 28$, although these multiplicities are not needed for the argument. $\square$

Theorem 10.5 (A nonsplit cyclic-eighteen family). For $t \in \mathbf{C} \setminus \{0, 1\}$ let C_t be the smooth projective model of

$$y^{18} = x(x-1)^4(x-t)^4,$$

and let B_t be its primitive $K = \mathbf{Q}(\zeta_{18})$ Jacobian factor. Then $\dim B_t = 6$, its $k = \mathbf{Q}(\sqrt{-3})$ action has signature $(3, 3)$, and its Weil plane $W_k(B_t)$ is algebraic for every t . The Hodge conjecture holds for every power of every B_t . For very general t , B_t is simple with $\text{End}^0(B_t) = K$, and

$$\text{Hg}(B_t) = \det_K^{-1}(\ker(N_{K/k} : U_K \rightarrow U_k)), \quad P_{B_t}(z) = (1 + z + z^2)^3 + 2z^3.$$

The inherited polarization has k -Hermitian determinant class -2 , not -1 . For these very general members no change of polarization makes this Weil structure split.

Proof. Put $\zeta = \zeta_{18}$ and orient the three K embeddings over one embedding of k by $j = 1, 7, 13$. The branch vector is $(1, 4, 4, 9)$; its entries are all nonzero modulo eighteen, so every primitive eigenspace has dimension two. The formula in [5, Lemma 2.7] gives their $H^{1,0}$ multiplicities $(2, 1, 0)$. Thus the three signatures are $(2, 0)$, $(1, 1)$, $(0, 2)$ and their k -sum is $(3, 3)$. The signed determinant cocharacter is $(2, 0, -2)$. Its cyclic Galois orbit spans the coordinate-sum-zero plane; hence its torus is exactly $\ker N_{K/k}$ for every t . In particular

$$T = \bigwedge_K^2 H^1(B_t, \mathbf{Q}) \subset H^2(B_t, \mathbf{Q})$$

has dimension six and no Tate subquotient. The displayed inclusion again means the sum of the same-embedding determinant lines, as in Lemma 13.12.

A fixed quotient with algebraic tensor classes. Write $B = B_t$, $C = C_t$, and let σ be the covering automorphism. The dihedral quotient of $C \times C$ by (σ, σ^{-1}) and interchange of the factors has field

$$a = x_1 + x_2, \quad b = x_1 x_2, \quad w = y_1 y_2, \quad w^{18} = b(b - a + 1)^4(b - ta + t^2)^4.$$

The right side has valuation one along $b = 0$, so this field has degree eighteen over $\mathbf{C}(a, b)$ and is the full invariant field. After homogenizing, its branch divisor consists of the four lines l_1, l_2, l_3, l_4 in Lemma 13.12, now with exponents $(1, 4, 4, 9)$. The same linear change of plane coordinates and a rescaling of w identify a resolution R_t birationally with

$$R : \quad W^{18} = XY^4Z^9(X + Y + Z)^4.$$

The Fermat surface $A^{18} + B^{18} + C^{18} + D^{18} = 0$ dominates this normalization by

$$[A : B : C : D] \longmapsto [A^{18} : B^{18} : C^{18} : AB^4C^9D^4].$$

As in Lemma 13.12, that Fermat surface is dominated by $F_{18} \times F_{18}$. Both maps are generically finite. Lemma 10.4, Abel–Jacobi pullback, the point blowup formula, and generically finite push–pull therefore prove the Hodge conjecture on every R_t^n . This uses the all-powers statement of that lemma, not only algebraicity on a single surface.

The sum Abel–Jacobi map $C^2 \rightarrow B$ pulls T back by antisymmetrization. This is fixed by the dihedral group, since interchange acts by minus the tensor flip on $H^1(C)^{\otimes 2}$. Transfer gives an algebraic injection $\alpha : T \rightarrow H^2(R_t)$. The algebraic determinant projector and orthogonality argument in Lemma 13.12 apply with three conjugate pairs instead of two. For an associated K -compatible polarization, the adjoint is now

$$\alpha^\dagger = p_T L_B^{-4} \alpha : H^2(R_t) \longrightarrow T, \quad L_B^{-4} : H^{10}(B) \longrightarrow H^2(B).$$

This inverse Lefschetz operator is algebraic. Both copies of T are primitive because they have no Tate quotient. Positivity and Cayley–Hamilton again give an algebraic left inverse $\beta = (\alpha^\dagger \alpha|_T)^{-1} \alpha^\dagger$. It follows from the Hodge conjecture for R_t^3 that every Hodge class in $T^{\otimes 3} \subset H^6(B^3)$ is algebraic. The cup-product map from $T^{\otimes 3}$ to $H^6(B)$ contains $W_k(B)$ in its image: over $\mathbf{C}$ each k -volume is the product of the three K -determinant lines above that embedding of k . Semisimplicity of polarizable rational Hodge structures lifts each Weil class to a Hodge class in $T^{\otimes 3}$. Applying the preceding algebraic correspondences and then the small diagonal pullback proves algebraicity of $W_k(B)$.

The group and the special parameters. The balanced eigenspace has connected monodromy $\mathrm{SU}(1, 1)$ by [5, Proposition 4.7(i)], so the generic fiber is not CM. More generally, at any non-CM parameter the derived projections to the three two-dimensional blocks must all be SL_2 ,

by Galois transitivity. A Galois-stable diagonal identification would identify all three factors, which is impossible since the adjoint Hodge circle is trivial on the pure blocks and nontrivial on the balanced block. Thus the derived group is SL_2^3 and the determinant computation gives the stated full preimage. Its commutant is K : the three standard blocks are distinct and their nontrivial determinant characters separate each block from its dual. This proves simplicity, the endomorphism assertion, and the displayed polynomial. Proposition 9.3 now gives algebraicity on all powers from the Weil plane just constructed.

At a CM parameter, if B is simple use Theorem 11.7. Otherwise the isotypic projectors commute with K , so each isotypic summand has dimension divisible by three. A single isotypic summand is a power of a simple variety of dimension at most three, whose powers are already covered by the low-dimensional inputs. If there are two summands, each has dimension three and is either a simple threefold or a cube of an elliptic curve. Its H^1 has K -rank one, so its endomorphisms commuting with K lie in K . A simple threefold summand therefore has CM field K and belongs to the unique primitive class X above; an elliptic summand has field k . The only nonisotypic possibility is consequently $X \times E^3$, whose powers are covered by [9, Corollary 1.11]. This proves the assertion at every allowed parameter without using a cycle-deformation principle.

The nonsplit determinant. Put $F = K^+$ and $\delta = 1 + 2\zeta^6 = \sqrt{-3}$. The geometric twisted-intersection matrix [10, Proposition 7.4] gives, up to a scalar in F and a K -basis change, the following skew-Hermitian form on the primitive rank-two space, with $x = \zeta, y = z = \zeta^4$:

$$G = \begin{pmatrix} \frac{1-xy}{(1-x)(1-y)} & \frac{1}{y-1} \\ \frac{y}{y-1} & \frac{1-yz}{(1-y)(1-z)} \end{pmatrix}, \quad \det G = \frac{1-xyz}{(1-x)(1-y)(1-z)}.$$

The matrix is the intersection pairing, not a form chosen merely to have the desired signature. One can use the balanced embedding $j = 7$ in the cited construction and Galois-conjugate back to this expression. Set $D = (1 - \zeta)(1 - \zeta^4)^2$. Then

$$D \in F, \quad N_{F/\mathbf{Q}}(D) = 3, \quad \det(G/\delta) = -\frac{2}{3D}, \quad N_{F/\mathbf{Q}} \det(G/\delta) = -\frac{8}{81}.$$

For $\alpha = \zeta + \zeta^{-1}$ one has $\alpha^3 - 3\alpha - 1 = 0$ and $D = 2\alpha^2 - 3\alpha - 2$, which also verifies the norm above. The trace Gram matrix in the basis $(1, \alpha, \alpha^2)$ is

$$\begin{pmatrix} 3 & 0 & 6 \\ 0 & 6 & 3 \\ 6 & 3 & 18 \end{pmatrix}, \quad \det = 81.$$

The determinant of the k -Hermitian trace form is therefore $81^2(-8/81) = -648 = -2 \cdot 18^2$. A conformal scalar changes the rank-two K -determinant by a square in F , and a basis change by a K/F norm; both become $k/\mathbf{Q}$ norms after taking the trace determinant. Thus the polarization class really is -2 . The prime 2 is inert in k , so 2 is not a $k/\mathbf{Q}$ norm; this class is not -1 . When $\mathrm{End}^0(B) = K$, every change of polarization multiplies the determinant by $N_{F/\mathbf{Q}}(a)^2$ for some totally positive $a \in F$. It cannot remove the obstruction. $\square$

The script `verify_cyclic18.py` checks the six eigenspace multiplicities, both rational maps, the Fermat character factors, and the intersection matrix and norms in $\mathbf{Q}[\zeta]/(\zeta^6 - \zeta^3 + 1)$. This explicit cyclic construction proves a specific nonsplit family. The complete row for all sextic fields and rational Hermitian forms instead follows from Theorem P.11 and the all-powers argument above, using the new external input.

11 S11: Type IV(6,1) (Proved)

Scope and status. Proved for every simple CM sixfold, on all powers, without a Galois hypothesis. Its Hodge rank is five or six. In rank five the sole extra generator is an algebraic split Weil plane. The finite character test and explicit Galois computations below provide additional certificates, not restrictions on the theorem.

Proposition 11.1. *Let A be a CM sixfold with field K of degree 12. If its Hodge group is the full norm-one torus $T_K = \ker(\text{Res}_{K/\mathbf{Q}} \mathbf{G}_m \rightarrow \text{Res}_{K^+/\mathbf{Q}} \mathbf{G}_m)$, then all Hodge classes on all powers are divisor-generated and $P_A(t) = (1+t)^6$. For a specified CM type, $\text{Hg}(A)$ is the smallest rational subtorus containing its Hodge circle; its cocharacter space is the rational span of the Galois translates of the signed CM-type vector in a normal closure.*

Proof. Over $\mathbf{C}$ the representation has six independent conjugate character pairs $\chi_i, -\chi_i$, each of multiplicity one. On powers a monomial is invariant under the full norm-one torus exactly when, for every i , the numbers of its positive and negative occurrences agree. It is then an exterior product of opposite-character degree-two pairings. By the (1,1) theorem and rational descent these are divisor-generated algebraic classes. On A one either chooses both members of a pair or neither, giving $\binom{6}{p}$ invariants in codimension p . The cocharacter assertion follows directly from the definition of the smallest rational torus: its cocharacter space must contain the Galois orbit and conversely the orbit span defines a rational subtorus after saturating the lattice. $\square$

Proposition 11.2 (An exact finite test for every specified CM type). *Let A be simple with $\text{End}^0(A) = K$ of degree twelve, and fix its CM type. Form a rational matrix M whose rows are all Galois translates of its signed CM-type vector in six conjugate-pair coordinates. For $\delta \in \{-1, 0, 1\}^6$ write $w(\delta)$ for its support size. Let n_j count the vectors with $M\delta = 0$ and $w(\delta) = j$. Then*

$$\dim B^2(A) = 15 + n_4, \quad \dim B^3(A) = 20 + 2n_4 + n_6.$$

The divisor-generated dimensions are 15 and 20 respectively. Thus $n_4 = n_6 = 0$ proves the Hodge conjecture for A even without a prior full-rank assertion about M .

Proof. The twelve complex eigenlines have characters x_i, x_i^{-1} . For an exterior monomial, let δ_i be the number of selected x_i coordinates minus the number of selected x_i^{-1} coordinates. The monomial is fixed by $\text{Hg}(A)$ exactly when $M\delta = 0$, by Proposition 11.1. A coordinate with $\delta_i = 0$ contributes either zero or two to exterior degree; a nonzero one contributes one. Consequently

$$\dim B^p(A) = \sum_{\substack{\delta \in \{-1, 0, 1\}^6 \\ M\delta = 0}} \binom{6 - w(\delta)}{p - w(\delta)/2},$$

with a binomial interpreted as zero unless its lower index is an integer between zero and its upper index. There are no weight-two nonzero relations: such a relation would identify two nonconjugate eigencharacters or make a conjugate character equal to another one, enlarging the commutant beyond $K \otimes \mathbf{C}$. The six opposite-coordinate pairings therefore form a basis of $B^1(A)_{\mathbf{C}}$. Their squarefree products span $D^p(A)_{\mathbf{C}}$ with dimension $\binom{6}{p}$. A weight-four relation contributes once in degree four and twice in degree six; a weight-six relation contributes once in degree six. This gives the formulas. Scalar extension preserves invariant dimensions and the image of the divisor multiplication maps, proving them over $\mathbf{Q}$. The last assertion follows from Lemma P.14. $\square$

Remark 11.3. This is an exhaustive computation for a specified field and CM type, using at most $3^6 = 729$ exponent vectors. A character-neutral exceptional monomial is not by itself an algebraic cycle. Nonzero n_4 or n_6 still requires a geometric algebraicity argument for its rational Galois-stable span. Theorem 11.7 below in fact shows that for a simple CM sixfold $n_4 = 0$ and $n_6 \in \{0, 2\}$; the test remains a useful independent check on the supplied CM data.

Corollary 11.4 (A checkable degenerate CM stratum). *In Proposition 11.2, suppose $n_4 = 0$ and every weight-six relation, up to sign, selects precisely the six embeddings above one embedding of an imaginary quadratic subfield $k \subset K$. Then all Hodge classes on A are algebraic; no subfield condition on K^+ is needed.*

Proof. The vanishing character says that the selected k -determinant is a Hodge class, equivalently that A has k -signature $(3, 3)$. Lemma P.18 makes its Weil plane algebraic. Apply this to every listed relation. Proposition 11.2 shows that these planes and divisors exhaust codimensions two and three. Their rational Galois-stable spans descend as in Lemma P.2, and Lemma P.14 finishes the proof. $\square$

Corollary 11.5 (All CM sixfolds of Hodge rank at least five). *If A is a simple CM sixfold and $\dim \text{Hg}(A) \geq 5$, the Hodge conjecture holds for every power of A .*

Proof. Write $K = \text{End}^0(A)$. If the dimension is six, $\text{Hg}(A) = U_K$ and all invariants on all powers are pairings, hence divisor-generated. If the dimension is five, [3, Lemma (3.7)] gives an imaginary quadratic $k \subset K$ with $\text{Hg}(A) = \ker(N_{K/k} : U_K \rightarrow U_k)$. The determinant is trivial on the Hodge circle, so the k -signature is $(3, 3)$. Lemma P.18 proves algebraicity of $W_k(A)$; Proposition 9.3 with $e = 6, q = 1$ proves the assertion on all powers. On A itself one has $n_4 = 0$, $n_6 = 2$, and $P_A(t) = (1 + t)^6 + 2t^3$. $\square$

Lemma 11.6 (A rank bound without a Galois hypothesis). *Every simple CM sixfold has Hodge rank at least five.*

Proof. Put $K = \text{End}^0(A)$, and let G be the Galois group of a normal closure of K . It acts by signed permutations on $\mathbf{Q}^6$, transitively on the twelve signed coordinate vectors; complex conjugation acts as -1 . Let $f \in \{1, -1\}^6$ be the signed CM type and $W = \langle Gf \rangle_{\mathbf{Q}}$. Then $r = \dim W = \dim \text{Hg}(A)$. The twelve weights of $H^1(A)_{\mathbf{C}}$ are distinct: equality of two would enlarge its Hodge commutant beyond $K \otimes \mathbf{C}$. Choose a basis of W from Gf . Each weight takes only the values $1, -1$ on these r vectors, so there are at most 2^r different weights. Hence $12 \leq 2^r$ and $r \geq 4$.

Suppose $r = 4$. Use the standard G -invariant positive definite dot product on $\mathbf{Q}^6$ and set $R = W^{\perp}$, of dimension two. Write $p_i = \text{pr}_R(e_i)$. None is zero, since transitivity would otherwise make them all zero. The distinct vectors in

$$\mathcal{P} = \{p_1, -p_1, \dots, p_6, -p_6\}$$

form one orbit of the finite rational orthogonal group Γ induced by G on R , and span R . Each occurs with the same multiplicity in this twelve-element multiset. Thus its cardinality m is an even divisor of twelve, and $m \neq 2$; consequently $m \in \{4, 6, 12\}$.

The rotation subgroup of a finite subgroup of $O(R_{\mathbf{R}})$ is cyclic. Since $-1 \in \Gamma$, its order n is even. A rational finite-order two-dimensional rotation has integral trace $2 \cos(2\pi/n)$; thus $n \in \{2, 4, 6\}$. The group has order n or $2n$. In particular, if m is six or twelve, Γ contains a rotation ρ of order three, with $1 + \rho + \rho^2 = 0$.

If $m = 4$, orient the six coordinates so that three projections equal u and three equal v , where u, v are independent. The relation

$$\sum_{i=1}^6 \varepsilon_i p_i = 0, \quad \varepsilon_i \in \{1, -1\}, \quad (*)$$

would require a sum of three signs to vanish. No such signed vector lies in W , contradicting $f \in W$.

If $m = 6$, after orienting and ordering coordinates the projections are $u, u, \rho u, \rho u, \rho^2 u, \rho^2 u$. The unique relation among the three distinct oriented vectors is $u + \rho u + \rho^2 u = 0$. Thus $(*)$ says

$$\varepsilon_1 + \varepsilon_2 = \varepsilon_3 + \varepsilon_4 = \varepsilon_5 + \varepsilon_6.$$

Either all three sums are zero, or all six signs are equal. The first possibility lies in the three-dimensional subspace V whose sums on the three pairs vanish; the second lies in the line $L = \mathbf{Q}(1, 1, 1, 1, 1, 1)$. Both are G -stable. Indeed G permutes the three pairs, and its signs on the three oriented projection vectors must all agree, by uniqueness of their relation. Consequently the orbit of f lies in V or in L , contradicting $\dim \langle Gf \rangle = 4$.

Finally let $m = 12$. The projection vectors are distinct even up to sign. The four orbits of $\langle \rho \rangle$ pair under negation, so orient coordinates to obtain

$$u, \rho u, \rho^2 u, \quad v, \rho v, \rho^2 v.$$

A signed sum of one triple is either zero, when all its signs agree, or is ± 2 times one of that triple's vectors. Nonzero sums from the two triples cannot cancel, since their vectors are distinct up to sign. Therefore every signed vector in W is constant on each triple. All of Gf lies in this two-dimensional subspace, again a contradiction. This excludes $r = 4$ and proves the lemma. $\square$

This recovers the degree-twelve rank conclusion in the Dodson–Zoller classification, summarized in [1, §2.2.2]. The rank in that CM-type convention is $\dim \text{MT}(A) = r + 1$. The proof above does not require the enumeration of the possible Galois groups.

Theorem 11.7 (All simple CM sixfolds). *For every simple CM abelian sixfold A and every $n \geq 1$, the rational Hodge conjecture holds for A^n . On A itself the only possible Hodge-dimension polynomials are*

$$P_A(t) = (1+t)^6 \quad \text{or} \quad (1+t)^6 + 2t^3.$$

In the first case all classes on every power are divisor-generated. In the second case divisors together with pullbacks of the algebraic Weil plane, under K -linear sum maps, generate them.

Proof. Lemma 11.6 gives Hodge rank five or six. Apply Corollary 11.5, including its invariant-generation argument on all powers. Its rank-five Weil plane is algebraic by the split trace polarization of Lemma P.18 and the sixfold theorem of Markman, not merely by the rank computation. $\square$

Remark 11.8 (Independent finite cross-check). The rank-four exclusion also has a small exact check. Normalize $f = (1, \dots, 1)$ and complete it to four independent signed orbit vectors. Distinct Hodge weights make the six columns distinct among the eight vectors $(1, a, b, c)$, $a, b, c \in \{1, -1\}$. The two omitted vertices have Hamming distance one, two, or three. Up

to cube symmetry these are all possibilities. For their row spaces, exact signed-permutation stabilizer computations give, respectively,

omitted distance	$ \text{Aut}_\pm(W) $	coordinate orbit sizes	$\dim\langle\text{Aut}_\pm(W)f\rangle$
1	96	6	3
2	32	$4 + 2$	4
3	96	6	1

The middle model is not transitive; the others cannot be generated by the orbit of f . The script `verify_cm_sixfold_rank.py` checks all $6!2^6$ signed permutations in each model using rational linear algebra. This verifies the rank lemma independently of listing Galois groups, but is not used in place of its proof.

Corollary 11.9 (All abelian Galois CM fields of degree twelve). *Let A be a simple CM sixfold. If $K = \text{End}^0(A)$ is Galois and abelian over $\mathbf{Q}$, the Hodge conjecture holds for every power of A .*

Proof. We show $\dim \text{Hg}(A) \geq 5$ and apply Corollary 11.5. For an abelian Galois CM field, a CM type is primitive precisely when its stabilizer under translation by $\text{Gal}(K/\mathbf{Q})$ is trivial: a nontrivial stabilizer would induce the type from its fixed CM subfield. The stabilizer cannot contain complex conjugation. The Hodge dimension is the number of nonzero odd Fourier components of the type; equivalently it is the dimension of the orbit span of its signed vector. This is also the character description in [12, §1], with the scalar component omitted when passing from MT to Hg.

The only abelian groups of order twelve are C_{12} and $C_2 \times C_6$. For $C_{12} = \langle \rho \rangle$, conjugation is ρ^6 . Write the signed vector as $f_0, \dots, f_5 \in \{\pm 1\}$ with $f_{j+6} = -f_j$, and put $F(T) = \sum_{j=0}^5 f_j T^j$. The rational odd-character factors are $T^6 + 1 = \Phi_4(T)\Phi_{12}(T)$, of dimensions two and four. The factor $\Phi_4 = T^2 + 1$ cannot divide F , since this would require $f_0 - f_2 + f_4 = f_1 - f_3 + f_5 = 0$, impossible for sums of three odd integers. If $\Phi_{12} = T^4 - T^2 + 1$ divided F , then

$$F(T) = (aT + b)(T^4 - T^2 + 1), \quad a, b \in \{\pm 1\}.$$

The full signed vector would be periodic with period four, contradicting primitivity. Thus the Hodge dimension is six.

For $C_2 \times C_6$, choose the first factor to be complex conjugation and a generator ρ of the complementary C_6 . The six signs f_j are arbitrary, and the rational Fourier factors of their cyclic orbit are $\Phi_1, \Phi_2, \Phi_3, \Phi_6$, of dimensions 1, 1, 2, 2. If Φ_3 divides F , the three sums $f_j + f_{j+3}$ are equal. If their common value is ± 2 , the type is constant; if it is zero, translation by $c\rho^3$ stabilizes it. Both violate primitivity. If Φ_6 divides F , the differences satisfy

$$(f_0 - f_3, f_1 - f_4, f_2 - f_5) = (t, -t, t), \quad t \in \{-2, 0, 2\}.$$

For $t = 0$ the vector has period three; for $t = \pm 2$ it has period two. Again primitivity is violated. Finally $F(1) = F(-1) = 0$ cannot both hold, because it would make each sum of three even-index or odd-index signs zero. Both two-dimensional components and at least one one-dimensional component therefore occur. This gives Hodge dimension at least five, as required. $\square$

Corollary 11.10 (All Galois CM fields of degree twelve). *Let A be a simple CM sixfold whose endomorphism field K is Galois over $\mathbf{Q}$. The Hodge conjecture holds for every power of A . If its Galois group is nonabelian, $\text{Hg}(A) = U_K$ and all these Hodge classes are divisor-generated.*

Proof. The abelian case is Corollary 11.9. Among the groups of order twelve, the nonabelian groups having a central involution are $C_2 \times S_3$ and

$$G = \langle a, b : a^3 = b^4 = 1, bab^{-1} = a^{-1} \rangle.$$

The other nonabelian group, A_4 , has trivial center and cannot have a central complex conjugation. In the second displayed group the conjugation is $c = b^2$; in the first it generates the C_2 factor. A primitive CM type has trivial right-translation stabilizer: a nontrivial stabilizer would make it induced from its fixed CM subfield. We prove that every such type has Hodge rank six.

For $C_2 \times S_3$, the signed type is a function $f : S_3 \rightarrow \{1, -1\}$. Its translates span a submodule of the rational regular representation

$$\mathbf{Q}[S_3] \simeq \mathbf{Q} \oplus \mathbf{Q} \oplus M_2(\mathbf{Q}).$$

Changing f to $-f$, assume its negative set S has size at most three. Size zero is imprimitive. If $|S| = 1$, the trivial and sign components of f are nonzero, and its matrix component is $-2\rho(s)$, hence invertible. Thus its translates span all six dimensions. If $S = \{s, t\}$ and $s^{-1}t$ has order two, right translation by $s^{-1}t$ fixes S , contradicting primitivity. Otherwise $s^{-1}t$ has order three, so s, t have the same parity. Both one-dimensional components are nonzero and $-2\rho(s)(1 + \rho(s^{-1}t))$ is invertible: the eigenvalues of $\rho(s^{-1}t)$ are the two primitive cube roots of unity. Again the rank is six.

If $|S| = 3$ and all its elements have the same parity, its type is imprimitive. Otherwise S has two elements of one parity and one of the other. Exactly two of the three right transpositions pair two elements inside S . The third transposition t therefore exchanges S with its complement. Right translation by ct fixes the full CM type, also contradicting primitivity. This exhausts the $C_2 \times S_3$ case.

For the second group the odd group algebra is

$$\mathbf{Q}[G]/(c+1) \simeq \mathbf{Q}(\sqrt{-1}) \oplus (-1, -3)\mathbf{Q}.$$

One verifies this directly by sending (a, b) to $(1, \sqrt{-1})$ in the first summand, and to $((-1 + j)/2, i)$ in the second, where $i^2 = -1, j^2 = -3, ij = -ji$. These are surjections onto distinct simple algebras and their dimensions sum to six. The quaternion summand is division, since it is definite over $\mathbf{R}$. Write the signed group-algebra element using the transversal $1, a, a^2, b, ab, a^2b$ as

$$x = \sum_{r=0}^2 u_r a^r + \sum_{r=0}^2 v_r a^r b, \quad u_r, v_r \in \{1, -1\}.$$

Its first component is $\sum u_r + \sqrt{-1} \sum v_r$, which is nonzero because both sums are odd. Its quaternion component vanishes only if each triple (u_0, u_1, u_2) and (v_0, v_1, v_2) is constant: use the direct sum $\mathbf{Q}(a) \oplus \mathbf{Q}(a)i$ and the sole relation $1 + a + a^2 = 0$. Such a type is invariant under right translation by a and is imprimitive. For a primitive type both division-algebra components are nonzero, hence invertible. Its translates span the whole six-dimensional odd algebra. This proves the asserted Hodge rank and, by Proposition 11.1, divisor generation on all powers. $\square$

Example 11.11 (Exact matrix certificate for the trace construction). Let $F = \mathbf{Q}(\theta)$ with $f(T) = T^6 + T^5 - 5T^4 - 4T^3 + 6T^2 + 3T - 1$; one can take $\theta = 2\cos(2\pi/13)$. This is the real sextic subfield of $\mathbf{Q}(\zeta_{13})$. For $K = \mathbf{Q}(i)F$ and $b' = 1/f'(\theta)$ the trace Hermitian form, in the $\mathbf{Q}(i)$ -basis

$1, \theta, \dots, \theta^5$, is

$$\left(\text{Tr}_{F/\mathbf{Q}} \frac{\theta^{u+v}}{f'(\theta)} \right)_{u,v=0}^5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 6 \\ 0 & 0 & 1 & -1 & 6 & -7 \\ 0 & 1 & -1 & 6 & -7 & 27 \\ 1 & -1 & 6 & -7 & 27 & -35 \end{pmatrix}.$$

Its determinant is -1 and its upper-left 3×3 block is zero. Choose the CM type by the positivity signs of $i/f'(\theta)$; after clearing denominators this trace form polarizes the associated CM abelian variety. The example verifies the split-form construction. For an arbitrary prescribed balanced CM type, the ordering step in Lemma P.18 is essential; this particular θ is not asserted to fit every type.

Example 11.12. Let C be the smooth projective curve $y^2 = x^{13} - 1$ and $A = J(C)$. There are 14 branch points of the hyperelliptic map including infinity, so $g(C) = 6$. The automorphism $(x, y) \mapsto (\zeta_{13}x, y)$ acts on the basis $x^{a-1}dx/y$, $1 \leq a \leq 6$, with eigenvalues ζ_{13}^a . It gives an embedding $K = \mathbf{Q}(\zeta_{13}) \hookrightarrow \text{End}^0(A)$ and CM type $\Phi = \{1, 2, 3, 4, 5, 6\}$ modulo 13. In the basis of six conjugate pairs, six Galois translates of the signed CM vector have matrix

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}, \quad \det(M) = -160.$$

The row with multiplier a has entry $+1$ at column b if $ab \bmod 13 \in \Phi$, and -1 otherwise; an inverse-multiplier convention gives the same orbit span. Exact elimination gives the displayed nonzero determinant, so the Hodge group is the full six-dimensional torus. Its rational centralizer is K (over $\mathbf{C}$ the twelve distinct character spaces have diagonal centralizer). Therefore $\text{End}^0(A) = K$ and A is simple. Proposition 11.1 applies, with dimensions $(1, 6, 15, 20, 15, 6, 1)$ and six rational divisor generators.

Remark 11.13. For a proper CM torus the vanishing of the total character must be tested on that smaller torus, not on T_K . Additional monomials may survive. Their exact degrees, rational dimensions, and algebraic representatives must be supplied before a degenerate type is counted as solved. Absolute Hodge classes are not thereby algebraic. Andre's reduction to split Weil classes also does not keep all auxiliary varieties in dimension six; see [13, Theorem 1.4].

Example 11.14 (A degenerate CM sixfold with algebraic exceptional classes). Let $F = \mathbf{Q}(\zeta_{13} + \zeta_{13}^{-1})$ and $K = F(i)$. Then $[K : \mathbf{Q}] = 12$ and $\text{Gal}(K/\mathbf{Q}) = C_6 \times C_2$, where complex conjugation is the second factor. Order the real embeddings of F cyclically under a generator of C_6 . Choose the CM type whose signs on the six embeddings of i are

$$f = (1, 1, -1, 1, -1, -1).$$

The complex torus $A = \mathbf{C}^6/\Phi(\mathcal{O}_K)$ is polarizable: choose $b \in F$ with prescribed signs by weak approximation so that ib defines a positive CM Riemann form. Clear denominators in

$$\psi(x, y) = \text{Tr}_{K/\mathbf{Q}}(ib x \bar{y}).$$

The signed CM matrix consists of the six cyclic shifts of f . Its rows sum to zero. A maximal minor is

$$\det \begin{pmatrix} 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 \\ -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 \end{pmatrix} = -16.$$

Thus its rational kernel is exactly $\mathbf{Q}(1, 1, 1, 1, 1)$ and its rank is five. All twelve restricted eigencharacters are distinct: their pairwise differences cannot be multiples of this all-ones vector. The commutant is therefore K , which proves $\text{End}^0(A) = K$ and simplicity. Its Hodge group is $\ker(N_{K/\mathbf{Q}(i)} : U_K \rightarrow U_{\mathbf{Q}(i)})$. The finite test gives $n_4 = 0$, $n_6 = 2$, and

$$P_A(t) = (1+t)^6 + 2t^3, \quad (1, 6, 15, 22, 15, 6, 1).$$

The real field F contains $\mathbf{Q}(\sqrt{13})$, so Lemma P.17 applied to $\mathbf{Q}(i, \sqrt{13})$ makes the two exceptional $\mathbf{Q}(i)$ -Weil classes algebraic. Divisors and these classes exhaust A ; Proposition 9.3 with $e = 6$, $q = 1$ additionally proves algebraicity on every power. This is a degenerate CM example, not merely a full-torus example.

Part II

The ten nonsimple rows

Product strata and the source of mixed classes

The following sections retain each original row rather than identifying an arbitrary product with its generic example. If a product $A \sim \prod B_i$ has the actual joint Hodge group $\prod \text{Sp}(H^1(B_i))$, then Lemma P.2 proves divisor generation on all its powers. For repeated factors, first collect isogeny classes; the group acts diagonally on their copies, and the invariant calculation includes Poincare divisors between those copies. If one already knows algebraicity on every power of the factors and has verified a product Hodge-group equality, the Kunnet invariant algebra is the tensor product of the factor invariant algebras, hence is algebraic. Neither product-group equality nor algebraicity on all powers follows merely from the Hodge conjecture for the individual factors. The symplectic examples use jointly very general tuples as in Proposition P.7. The CM and unitary examples verify their different joint groups separately.

12 N1: The $5 + 1$ case (Proved)

Scope and status. Proved for every sixfold in the $5 + 1$ row. The all-powers strengthening is not asserted for the entire row.

Theorem 12.1. *Let $A \sim B \times E$ with B a simple fivefold and E elliptic. Every rational Hodge class on A is algebraic. Its Hodge ring is generated by divisors, except that a rational two-dimensional Weil space in codimension three must be added if E has CM by a field acting on B with signature $(3, 2)$, up to interchange.*

Proof. Apply Lemma P.19 with $g = 5$ and isogeny invariance. In the exceptional case the k -action on the elliptic factor is oriented so the combined signature is $(3, 3)$. Its rank-one summand permits Lemma P.16, hence Markman's discriminant—1 theorem. This step proves the algebraicity of the mixed classes, rather than inferring it from the individual factors. $\square$

Example 12.2. Choose $(B_1, \dots, B_2)$ jointly very generally in $\prod_i \mathcal{A}_{d_i}(N)$ with $d_i = (5, 1)$, as constructed in Proposition P.7. Then each factor is simple with $\text{End}^0(B_i) = \mathbf{Q}$, and the factors are pairwise nonisogenous. Their product is an actual sixfold of this partition, with $\text{End}^0(A) = \mathbf{Q}^2$ and the proved joint group above. The Hodge-space dimensions in codimensions 0 through 6 are $(1, 2, 2, 2, 2, 1)$. An explicit monomial basis over $\mathbf{Q}$ on A is $\theta_1^{a_1} \cdots \theta_2^{a_2}$ with $0 \leq a_i \leq d_i$; every such monomial is an intersection of divisors. On powers add Poincare divisors between the copies of each individual B_i . There are no cross-factor divisors or higher mixed invariants in this verified jointly generic example.

Example 12.3 (A genuinely exceptional $5 + 1$ product). Let $k = \mathbf{Q}(i)$ and take a very general polarized fivefold B with k -Hermitian datum $(k^5, \text{diag}(1, 1, 1, -1, -1))$. The unitary PEL domain has dimension six; its dense derived monodromy and nonzero determinant circle give $\text{Hg}(B) = U(3, 2)$ and $\text{End}^0(B) = k$. Thus B is simple. Let $E : y^2 = x^3 - x$ with the opposite k -action. Then $A = B \times E$ has signature $(3, 3)$ and determinant -1 . Lemma P.19 gives

$$P_A(t) = (1 + t + \cdots + t^5)(1 + t) + 2t^3, \quad (\dim B^p(A))_{p=0}^6 = (1, 2, 2, 4, 2, 2, 1).$$

Products of the two polarization classes span the divisor part; the remaining middle plane is $W_k(A)$, algebraic by the split Weil theorem. This example explicitly exhibits the mixed classes absent from the jointly symplectic example.

13 N2: The $4 + 2$ case (Proved)

Scope and status. All cases with a CM fourfold are now covered, using the new external CM theorem for the CM–CM products and the earlier independent strata for a non-CM surface. The other previously proved strata and weighted covering families remain covered. For the residual non-CM fourfold with primitive quartic CM field and a matching CM surface, Theorem 13.19 constructs a nonzero mixed cycle from the auxiliary-spin and CM inputs. This completes the sixfold row; the residual models also satisfy the all-powers conclusion.

Corollary 13.1 (All CM–CM products in N2). Assume the CM input in Theorem P.9. If B is a simple CM fourfold and S a simple CM surface, then the Hodge conjecture holds on every power of $B \times S$. In particular every coupled case in Proposition 13.8 is algebraic; no independence of the two Hodge groups is required. Together with Proposition 13.4, this settles every sixfold in N2 whose fourfold factor is CM.

Proof. The product $B \times S$ is CM, even when its Hodge torus is a proper subgroup of $\text{Hg}(B) \times \text{Hg}(S)$. Lemma P.12 applies to each of its powers. If instead S is not CM, Proposition 13.4 already proves the Hodge conjecture for the sixfold. These two alternatives exhaust surfaces and prove the final assertion. $\square$

Example 13.2 (A CM fourfold times a CM surface). Let $L = \mathbf{Q}(\zeta_{15})$ and take the CM type with embedding exponents $\Phi = \{1, 2, 4, 7\}$ modulo 15. It selects one embedding from each conjugate pair. Its multiplicative stabilizer in $(\mathbf{Z}/15\mathbf{Z})^\times$ is trivial: any stabilizing element belongs to Φ , and direct multiplication excludes 2, 4, 7. Since $L/\mathbf{Q}$ is Galois, the type is primitive. The standard CM construction therefore gives a simple fourfold $B = \mathbf{C}^4/\Phi(\mathcal{O}_L)$. For its polarization, choose $\delta \in L$ with $\bar{\delta} = -\delta$ and $\text{Im } \phi(\delta) > 0$ for every $\phi \in \Phi$, by weak approximation. An integral multiple of $\text{Tr}_{L/\mathbf{Q}}(\delta x \bar{y})$ on $\mathcal{O}_L$ is a Riemann form. Primitivity of the CM type gives simplicity. Take $S = J(y^2 = x^5 - 1)$, the simple CM surface with field $\mathbf{Q}(\zeta_5)$. Corollary 13.1 proves the Hodge conjecture for every $(B \times S)^n$ by the external CM theorem. This argument does not require computing the joint torus or assuming that its exceptional classes are divisor-generated.

Proposition 13.3. *Let $A \sim B_1 \times \cdots \times B_2$ with dimensions $(4, 2)$. Assume the actual joint Hodge group is the independent product of the full symplectic groups on these factors. Then all Hodge classes on all powers of A are divisor-generated, and $P_A(t) = (1 + t + \cdots + t^4)(1 + t + t^2)$.*

Proof. Use Lemma P.1 to replace the isogenous variety by the product. Under the stated group equality,

$$\left(\bigwedge \bigoplus_i H^1(B_i)^{\oplus n} \right)^{\text{Hg}(A)} = \bigotimes_i \left(\bigwedge H^1(B_i)^{\oplus n} \right)^{\text{Sp}(H^1(B_i))}.$$

Each factor is generated by degree-two contractions by Lemma P.2. These are divisor classes on the product, so every invariant is algebraic. On one copy the generators are the polarization powers through the dimension of each factor, giving the displayed polynomial. $\square$

Proposition 13.4 (Two further complete strata). *Let $A = B \times S$, with B a simple fourfold and S a simple surface. The Hodge conjecture holds for A if either S is not CM, or the center of $\text{End}^0(B)$ is totally real or imaginary quadratic.*

Proof. If S is not CM, [3, Proposition (2.4), Lemma (3.4)] and $\text{Hom}(S, B) = 0$ give $\text{Hg}(A) = \text{Hg}(B) \times \text{Hg}(S)$. If S is CM, its Hodge group is a $\mathbf{Q}$ -simple rank-two torus. The connected center of $\text{Hg}(B)$ embeds in the norm-one torus of the CM center of $\text{End}^0(B)$; it is trivial for a totally real center and has rank at most one for an imaginary quadratic center. Goursat's lemma therefore excludes a common nontrivial quotient with $\text{Hg}(S)$, proving the same product equality. The exterior Kunneth invariants are consequently tensor products of factor Hodge classes. The fourfold theorem in [11, Corollary 1.6.1] and the prime-dimensional surface theorem make these factor classes algebraic. Their external products prove the assertion. $\square$

Proposition 13.5 (A splitting-field separation test). *Let $A = B \times S$, where B is a simple fourfold and S a simple CM surface with field F . Suppose the center K of $\text{End}^0(B)$ is CM. Let $L_F, L_K \subset \overline{\mathbf{Q}}$ be their normal closures. If $L_F \not\subset L_K$, then $\text{Hg}(A) = \text{Hg}(B) \times \text{Hg}(S)$ and the Hodge conjecture holds for A . In particular this holds when $K/\mathbf{Q}$ is multiquadratic, including every biquadratic quartic center.*

Proof. The torus $\text{Hg}(S) = U_F$ is $\mathbf{Q}$ -simple of rank two, with splitting field L_F and Galois group C_4 or D_4 . The connected center Z of $\text{Hg}(B)$ embeds into U_K and is therefore split by L_K . By Goursat, any failure of independence would make U_F an isogeny quotient of Z . Its rational character module would then be split by L_K , forcing $L_F \subset L_K$. This contradiction gives independence; the fourfold and surface theorems imply algebraicity of the tensor-product Hodge ring. If $K/\mathbf{Q}$ is multiquadratic, every quotient of its Galois group has exponent at most two, and thus cannot be C_4 or D_4 . $\square$

Proposition 13.6 (All residual non-CM-fourfold groups in N2). *Let $A = B \times S$ be an N2 case with B not CM and not already covered by Propositions 13.4–13.5. Then $\text{End}^0(B) = K$ is a primitive quartic CM field, $\text{End}^0(S) = F$ is a primitive quartic CM field, and the two K -signatures on B are $(2, 0), (1, 1)$ after orientation. In particular $\text{Hg}(B)$ is the full K -unitary group. Orient the two complex coordinates on U_F at the CM type of S . There are precisely two alternatives:*

1. *There is no rational torus morphism $v : U_F \rightarrow U_K$ whose complex coordinate expression is $(u, v) \mapsto (uv, u/v)$, up to conjugating the second K embedding. Then the joint group is the independent product and the Hodge conjecture holds.*
2. *Such a morphism exists. It is an isogeny, and the joint group is*

$$\{(g, s) \in \text{Hg}(B) \times U_F : \det_K g = v(s)\}.$$

Its Hodge ring and polynomial are exactly those in Lemma 13.10. Algebraicity is equivalent to the existence of one codimension-two algebraic cycle with nonzero projection to the four-dimensional space Q in that lemma.

The torus-morphism condition is a finite Galois-equivariance test on the displayed integral matrix; it is not an algebraicity assumption.

Proof. The previous strata leave a larger CM center on the non-CM fourfold and a simple CM surface. The dimension formula forces the fourfold's endomorphism algebra to be a quartic field K ; a degree-eight field would make B CM. Biquadratic K was already separated by Proposition 13.5, so K is primitive. Two pure signatures would make the Hodge circle, and hence its rational group, central and would make B CM. Two balanced signatures would put the group in SL_2^2 ; the standard–dual intertwiners would contradict the endomorphism commutant K . Only the stated mixed signatures remain. The derived factors are SL_2^2 , not a diagonal group, because only one block has nontrivial adjoint Hodge circle. The determinant cocharacter and its conjugates span U_K , giving the full group.

Both U_K and U_F are rationally simple two-dimensional tori. If the joint group is not independent, Goursat gives an isogeny between these determinant tori. On rational cocharacter spaces it is a Galois-equivariant isomorphism f sending $(1, 1)$ to $(2, 0)$. The Galois actions are irreducible signed permutation actions of C_4 or D_4 on a plane. Their invariant positive quadratic forms are unique up to a scalar, so f is a similarity for the standard Euclidean forms, with multiplier 2. Orthogonality to $(1, 1)$ consequently gives

$$f(1, -1) = (0, \pm 2), \quad f = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{up to the stated conjugation.}$$

This integral Galois-equivariant matrix gives the claimed rational torus morphism. Conversely, if it exists, the displayed group is connected, has dimension $\dim \text{Hg}(B)$, and contains the Hodge circle. Surjectivity of the actual group onto $\text{Hg}(B)$ proves equality. If it does not exist, Goursat forces independence; the factor Hodge classes are already algebraic.

In the coupled case Lemma 13.10 leaves exactly Q . Its projection is algebraic: use the determinant projector on $H^2(B)$ constructed in Lemma 13.12, and the degree-two projector on S . For a general $a \in F$, pullback by a has four distinct eigenvalues on the four mixed character lines of $H^2(S)$, namely the products obtained by choosing one embedding from each conjugate pair. The Galois group acts transitively on these four products, so their polynomial over $\mathbf{Q}$ is irreducible of degree four. The same four eigenvalues occur once each on Q . Thus the rational algebra generated by this algebraic operator makes Q a one-dimensional vector space over a quartic field. Every nonzero algebraic projection generates all of Q by algebraic correspondences, proving the criterion. $\square$

Proposition 13.7 (Rank-three CM fourfolds are independent of surfaces). *Let B be a simple CM fourfold with $L = \text{End}^0(B)$. Its Hodge rank is three or four. In rank three, $\text{Hg}(B)$ is a $\mathbf{Q}$ -simple torus. Consequently, for every simple CM surface S ,*

$$\text{Hg}(B \times S) = \text{Hg}(B) \times \text{Hg}(S),$$

and the Hodge conjecture holds for $B \times S$. In this case

$$P_{B \times S}(t) = (1 + t)^6 + 2t^2(1 + t)^2 = 1 + 6t + 17t^2 + 24t^3 + 17t^4 + 6t^5 + t^6.$$

Proof. The eight weights of $H^1(B)_{\mathbf{C}}$ are distinct because its Hodge commutant is $L \otimes \mathbf{C}$. Evaluating them on a basis of signed CM-type orbit vectors gives $8 \leq 2^r$, where $r = \dim \text{Hg}(B)$,

so $3 \leq r \leq 4$. Suppose $r = 3$. Lemma (3.7) of [3] gives an imaginary quadratic $k \subset L$ with $\text{Hg}(B) = \ker(N_{L/k} : U_L \rightarrow U_k)$. Orient the four embedding coordinates over one embedding of k . The signed CM vector has two positive and two negative entries, and the Galois action is by permutations with a common sign. Its permutation image $P \subset S_4$ is transitive. On the hyperplane $\sum x_i = 0$ the action is the standard permutation representation, possibly twisted by a one-dimensional character.

The transitive subgroups of S_4 are C_4, V_4, D_4, A_4, S_4 . For the first three, an orbit of a vector with two positive and two negative entries spans at most a plane: in the square model the two positive positions are either adjacent or opposite. The V_4 case is a subgroup of this same model. Thus rank three forces $P = A_4$ or S_4 . Both are doubly transitive, so their standard three-dimensional representation is irreducible over $\mathbf{Q}$, as is its character twist. This proves rational simplicity. The rank-two torus $\text{Hg}(S)$ is also rationally simple and cannot be a quotient of this rank-three torus. Goursat gives independence. The Hodge ring is the tensor product of the factor Hodge rings; the fourfold theorem of [11] and the surface theorem give its algebraicity. For the norm-kernel fourfold, the only non-pairing weights on one copy are the two k -volumes. Thus $P_B(t) = (1+t)^4 + 2t^2$; multiplication by $P_S(t) = (1+t)^2$ gives the stated polynomial. $\square$

Proposition 13.8 (All CM-fourfold couplings in N2). *Let B be a simple CM fourfold, $L = \text{End}^0(B)$, and let S be a simple CM surface with field F . If their joint Hodge group is not the independent product, then $\text{Hg}(B) = U_L$ and there is a primitive quartic CM subfield $K \subset L$ such that the K -signatures on B are $(2,0), (1,1)$ after orientation. There is a rational torus isogeny $v : U_F \rightarrow U_K$ with complex coordinates $(u,v) \mapsto (uv, u/v)$, and the actual joint group is*

$$H = \{(b,s) \in U_L \times U_F : N_{L/K}(b) = v(s)\}.$$

Conversely, these field, signature, and equivariant-isogeny data with $\text{Hg}(B) = U_L$ imply this group equality. Put

$$T = \bigwedge_{\mathbf{K}}^2 H^1(B, \mathbf{Q}), \quad Q = (T \otimes H^2(S, \mathbf{Q})) \cap H^{2,2}(B \times S),$$

using the rational-summand convention of Lemma 13.12. Then $\dim_{\mathbf{Q}} Q = 4$, divisors and Q generate the Hodge ring, and

$$P_{B \times S}(t) = (1+t)^6 + 4t^2(1+t)^2 = 1 + 6t + 19t^2 + 28t^3 + 19t^4 + 6t^5 + t^6.$$

The Hodge conjecture is equivalent to the existence of a codimension-two algebraic cycle whose projection to Q is nonzero.

Proof. Proposition 13.7 leaves only $\text{Hg}(B) = U_L$. Since U_F is rationally simple of rank two, a non-independent joint torus projects isogenously onto U_L . On rational cocharacter spaces it is the graph of a surjection

$$f : \mathbf{Q}^4 \longrightarrow \mathbf{Q}^2, \quad f(h_B) = h_S = (1,1), \quad h_B \in \{1, -1\}^4.$$

This map is Galois-equivariant. Write $a_i = f(e_i)$. Transitivity on the eight signed L embeddings shows that all a_i are nonzero and that the multiset $\{\pm a_i\}_{i=1}^4$ is one orbit, each vector occurring with equal multiplicity. The Galois image on the target is the standard signed-plane group $\Gamma = C_4$ or D_4 . Its nonzero orbits have four or eight elements, so there are exactly two possibilities.

If all eight vectors are distinct, equivariance identifies the signed permutation action on the source with the regular action of D_4 on its eight elements; an element acting trivially on the target must fix every signed source coordinate. The source is therefore the negative-central part of the regular module. Writing

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

identifies this four-dimensional module with $M_2(\mathbf{Q})$ under left multiplication. A signed CM vector corresponds to

$$M = \epsilon_1 I + \epsilon_2 R + \epsilon_3 D + \epsilon_4 RD = \begin{pmatrix} \epsilon_1 + \epsilon_3 & -\epsilon_2 + \epsilon_4 \\ \epsilon_2 + \epsilon_4 & \epsilon_1 - \epsilon_3 \end{pmatrix}, \quad \epsilon_i \in \{1, -1\}.$$

Its determinant is $\epsilon_1^2 + \epsilon_2^2 - \epsilon_3^2 - \epsilon_4^2 = 0$, so it has rank one. Its left orbit consequently spans only two dimensions, contradicting $\langle Gh_B \rangle = \mathbf{Q}^4$.

There are thus four distinct projection vectors, each repeated twice. Orient and reorder the coordinates so that

$$(a_1, a_2, a_3, a_4) = (a, a, b, b), \quad b \perp a, \quad |b| = |a|.$$

The orbit is a square. The image of a signed vector is $n_1 a + n_2 b$, where $n_i \in \{-2, 0, 2\}$ are its pair sums. If both sums are nonzero, that signed vector is constant on each pair. The pair-constant plane is Galois-stable, contradicting full Hodge rank. Both sums cannot be zero because $f(h_B) = (1, 1)$. Hence, after changing orientations,

$$h_B = (1, 1, 1, -1), \quad a = (1/2, 1/2), \quad b = (1/2, -1/2).$$

The four fibers of the map from signed embeddings to these projection vectors form a Galois-stable block system. It defines a degree-four subfield $K \subset L$. Conjugation interchanges opposite blocks, so K is CM; the associated rank-two signed module is irreducible, so K is primitive. The displayed CM vector gives its signatures $(2, 0), (1, 1)$.

The norm $N_{L/K}$ is the pair-sum map on cocharacters. If $\nu_* = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, then $\nu_* f = (N_{L/K})_*$. Surjectivity of f proves that ν_* is Galois-equivariant. It is integral in these standard cocharacter lattices and therefore defines the asserted torus isogeny. Conversely, the displayed group H is connected because $\ker(N_{L/K})$ is connected: its complex coordinate map is $(x_1, x_2, y_1, y_2) \mapsto (x_1 x_2, y_1 y_2)$. It contains the joint Hodge circle and has dimension four. Surjectivity of the actual group onto U_L gives equality.

For the invariants, split each two-dimensional block in Lemma 13.10 into its two independent CM characters. The determinant-kernel invariants in one block are its two opposite-character pairings, their product, and its two volume lines. The same neutrality equations allow only one active volume block. The pairings give $(1+t)^6$, and the four mixed degree-four lines contribute $4t^2(1+t)^2$. They form Q over $\mathbf{Q}$ and generate the quotient by the divisor algebra. The determinant projector on B and the degree-two projector on S are algebraic. On Q , a general $a \in F$ has four distinct eigenvalues in one Galois orbit, the four mixed products of embeddings of F . Its minimal polynomial is irreducible of degree four. Thus Q is rank one over the quartic field generated by this algebraic operator. Every nonzero algebraic projection generates all of Q , proving the final criterion. $\square$

The script `verify_n2_cm_structure.py` checks all thirty subgroups of S_4 , all sixteen signed vectors in the regular dihedral model, and the pair-sum alternatives. These finite checks supplement the preceding proofs; they assert no cycle existence. Together with Proposition 13.6, the result leaves only the single mixed-cycle construction in each residual N2 case, not an unclassified common torus quotient.

Remark 13.9 (A finite-kernel obstruction to an automatic reduction). In either coupled N2 model, projection of the joint group onto $\text{Hg}(B)$ has kernel μ_2 : its nontrivial element acts trivially on $H^1(B)$ and as -1 on $H^1(S)$. Thus $H^1(S)$ is not an object of the tensor category generated by $H^1(B)$ and Tate structures. Even an all-powers theorem for B does not make S an available factor by tensor operations. The even-degree mixed Hodge tensors still require an algebraic correspondence; the projection criterion above records exactly that missing step.

Lemma 13.10 (The quartic mixed invariant calculation). *Let B be a fourfold and P an abelian surface. Suppose that over $\mathbf{C}$ their actual joint Hodge representation is*

$$H^1(B)_{\mathbf{C}} = V_1 \oplus V_1^* \oplus V_2 \oplus V_2^*, \quad H^1(P)_{\mathbf{C}} = \mathbf{C}_u \oplus \mathbf{C}_{u^{-1}} \oplus \mathbf{C}_v \oplus \mathbf{C}_{v^{-1}}, \quad \dim V_j = 2,$$

with group

$$H = \{(g_1, g_2, u, v) : \det g_1 = uv, \det g_2 = u/v\} \subset \mathrm{GL}_2^2 \times \mathbf{G}_m^2.$$

Suppose also that the sum $T_{\mathbf{C}}$ of the four determinant lines $\wedge^2 V_j, \wedge^2 V_j^*$ descends to $T \subset H^2(B, \mathbf{Q})$. There is a four-dimensional rational space

$$Q = (T \otimes H^2(P, \mathbf{Q})) \cap H^{2,2}(B \times P) \subset B^2(B \times P)$$

such that the Hodge ring is generated by divisors and Q . More precisely,

$$P_{B \times P}(t) = (1 + t + t^2)^2(1 + t)^2 + 4t^2(1 + t + t^2) = 1 + 4t + 12t^2 + 14t^3 + 12t^4 + 4t^5 + t^6.$$

Thus the Hodge conjecture on this sixfold is equivalent to the algebraicity of Q , not merely of a middle-degree Weil plane.

Proof. The derived group is SL_2^2 . On $\wedge^\bullet(V_j \oplus V_j^*)$ its invariants are the pairing powers $1, \theta_j, \theta_j^2$ of weight zero and the two degree-two volumes of weights d_j, d_j^{-1} . A pairing times a volume in the same block is zero. Write $m_j \in \{-1, 0, 1\}$ for the chosen determinant exponent. The required P weights are $-(m_1 + m_2)$ and $-(m_1 - m_2)$, each of which can only be $-1, 0, 1$ on a single copy of P . Therefore both m_j cannot be nonzero. When both vanish the invariants are exactly the pairing monomials, with polynomial $(1 + t + t^2)^2(1 + t)^2$. Otherwise there are four possibilities: one B volume and one coordinate of each of the u and v pairs. They give the four lines of $Q_{\mathbf{C}}$ in degree four. Each can only be multiplied by $1, \theta_{3-j}, \theta_{3-j}^2$ from the inactive B block, giving $4t^2(1 + t + t^2)$. Rational descent of the surjective multiplication map $D^\bullet \otimes Q \rightarrow B^\bullet/D^\bullet$ proves the claimed generation over $\mathbf{Q}$. $\square$

Proposition 13.11 (Tensor contraction for every residual mixed model). *Let $A = B \times P$ be a residual coupled case of N2 or N3 in Propositions 13.6, 13.8, or 14.6; thus B is simple and P is a CM surface or a product of the two specified CM elliptic curves. The following are equivalent:*

1. *some codimension-two algebraic cycle on A has nonzero projection to the four-dimensional mixed space Q ;*
2. *the Hodge conjecture holds for A ;*
3. *the Hodge conjecture holds for A^n for every $n \geq 1$.*

Proof. The single-cycle criteria in the cited propositions give (1) $\iff$ (2), and (3) $\implies$ (2) is immediate. It remains to prove (1) $\implies$ (3). Those criteria first make all of Q algebraic using field endomorphisms; fix this input.

We work in tensor powers of $H^1(A)_{\mathbf{C}}$; alternation will recover cohomology of arbitrary powers. Kunnet projectors, factor permutations, endomorphisms, cup products, and polarization contractions on abelian varieties are algebraic correspondences. In particular an algebraic class in $Q \subset H^2(B) \otimes H^2(P)$ can be separated into four H^1 slots by addition-map pullback and degree-one projectors. Inverse polarization contractions are algebraic by the Lefschetz standard conjecture for abelian varieties. We may therefore use these tensor operations on algebraic classes.

First take invariants under $D = \ker(\det_K : \mathrm{Hg}(B) \rightarrow U_K)$. For non-CM B , $D_{\mathbf{C}} = \mathrm{SL}_2^2$. Its tensor invariants are generated by evaluation pairings and the two block volumes and their

duals. For CM B , $D_C = \mathbf{G}_m^2$: in each rank-two K block its characters are z, z^{-1} , with the two dual characters on the dual block. Pairwise opposite-character tensors are exactly the individual evaluation pairings, the block volume, and its dual. In this CM case the endomorphisms in L separate the two lines: applying their eigenspace projectors to individual slots of the antisymmetric volume gives both ordered opposite-character tensors. With these algebraic projectors the listed tensors again generate all tensor invariants. The evaluation pairings in either case are rational divisor classes after taking the full rational span. An opposite pair of volumes in the same block can be replaced by the determinant of evaluation pairings. Thus it suffices to handle volume monomials with signed exponents $r, s \in \mathbf{Z}$ for the two blocks.

The remaining torus has coordinates u, v on $H^1(P)$, and the block volumes have characters $uv, u/v$. Neutrality requires residual P character $u^{-(r+s)}v^{-(r-s)}$. A positive or negative block-one volume together with one coordinate of each of the u, v pairs is a line of Q_C ; the analogous block-two tensors give the other two lines. Take $|r| + |s|$ appropriate signed copies of these algebraic Q tensors. Their B part is the desired volume monomial. Their P part has the correct total character, but may have extra opposite u or v slots. Contract every such opposite pair with the polarization. Each character line is one-dimensional and its pairing with the opposite line is nonzero, so this yields a nonzero multiple of the required residual tensor. Any additional opposite slots in a given target tensor are supplied by divisor pairings.

This constructs every torus-neutral tensor from Q and algebraic pairings, even when it is not an ordinary exterior product of classes pulled back from A . Character projections and tensor permutations may be performed after scalar extension: endomorphism eigenspace projectors are complex linear combinations of rational algebraic correspondences. Consequently the complex span of rational algebraic classes contains every invariant tensor. Alternating the slots into any prescribed copies of A , and then descending the surjective rational cycle-class span, proves the assertion on every A^n . No implication from $\mathrm{HC}(A)$ to all powers is used outside these explicitly verified group models. $\square$

Lemma 13.12 (A Fermat correspondence for weighted cyclic covers). *Let $m \in \{5, 8, 10, 12\}$ and let $\mathbf{a} = (a_1, a_2, a_3, a_4)$ satisfy $1 \leq a_i < m$, $\sum a_i \equiv 0 \pmod{m}$, and $\gcd(m, a_1, a_2, a_3, a_4) = 1$. Assume the four primitive multiplicities*

$$d_j = -1 + \sum_{i=1}^4 \left\langle -\frac{ja_i}{m} \right\rangle, \quad j \in (\mathbf{Z}/m\mathbf{Z})^\times,$$

have multiset $\{2, 1, 1, 0\}$. For $t \in \mathbf{C} \setminus \{0, 1\}$ let $C_{m,\mathbf{a},t}$ be the smooth projective model of

$$y^m = x^{a_1}(x-1)^{a_2}(x-t)^{a_3},$$

and let $B_{m,\mathbf{a},t}$ be the primitive $K = \mathbf{Q}(\zeta_m)$ isogeny factor of its Jacobian. This factor has dimension four; the entire Jacobian need not have dimension four. Put

$$T = \bigwedge_K^2 H^1(B_{m,\mathbf{a},t}, \mathbf{Q}) \subset H^2(B_{m,\mathbf{a},t}, \mathbf{Q}),$$

where the inclusion denotes the rational summand comprising the four same-embedding determinant lines. For every abelian surface P , all rational Hodge classes in $T \otimes H^2(P, \mathbf{Q})$ are algebraic on $B_{m,\mathbf{a},t} \times P$, for every allowed t . For $m = 5, 8, 12$ and $\mathbf{a} = (1, 1, 1, m-3)$ retain the shorthand $C_{m,t}, B_{m,t}$ used below.

Proof. Write $B = B_{m,\mathbf{a},t}$ and $C = C_{m,\mathbf{a},t}$. Connectedness follows from the gcd assumption. Since every primitive character is nontrivial at each of the four branch points, its H^1 eigenspace has dimension two. The displayed multiplicity formula is [5, Lemma 2.7]. Orient the conjugate pairs so that the pure block precedes the balanced block. The determinant cocharacter $(2, 0)$ and

its Galois conjugates span U_K . Consequently the determinant image of $\text{Hg}(B)$ is U_K at every parameter, even when extra endomorphisms appear. In particular T has no Tate subquotient and is primitive for any polarization of B .

We also record a group dichotomy used below. Each derived projection to a two-dimensional block is either trivial or SL_2 , and Galois transitivity makes the two projections simultaneous. In the nontrivial case a diagonal identification is impossible: the adjoint Hodge circle is trivial on the pure-signature block and nontrivial on the balanced block. Thus either B is CM, or $\text{Hg}(B)_{\mathbb{C}} = \text{GL}_2^2$ and $\text{End}^0(B) = K$. This argument only uses the signatures and works for any quartic CM field with these signatures. No simplicity or endomorphism-equality assumption is needed for the correspondence constructed here.

The quotient surface. Let $\sigma(x, y) = (x, \zeta_m y)$. On $C \times C$, the automorphisms $r = (\sigma, \sigma^{-1})$ and $s(p, q) = (q, p)$ generate a dihedral group of order $2m$. Its invariant function field has coordinates

$$a = x_1 + x_2, \quad b = x_1 x_2, \quad w = y_1 y_2, \quad w^m = b^{a_1} (b - a + 1)^{a_2} (b - ta + t^2)^{a_3}.$$

Indeed these functions are invariant. The Kummer extension they define has degree m : the valuations along the three displayed lines, together with the line at infinity, are a_1, a_2, a_3, a_4 modulo m , whose common gcd with m is one. The function field $\mathbb{C}(C \times C)$ has degree $2m^2$ over $\mathbb{C}(a, b)$, so the stated field is the full dihedral invariant field. Let R_t be a smooth projective resolution of the quotient. Homogenizing with z gives a cyclic cover of $\mathbb{P}^2$ branched on the four lines

$$l_1 = b, \quad l_2 = b - a + z, \quad l_3 = b - ta + t^2 z, \quad l_4 = z, \quad w^m = l_1^{a_1} l_2^{a_2} l_3^{a_3} l_4^{a_4}.$$

Set $X = (1 - t)l_1$, $Y = tl_2$, $Z = t(t - 1)l_4$. Then $l_3 = X + Y + Z$. After rescaling w by a nonzero complex constant, R_t is birational to the normalization of

$$R_{m,a} : \quad W^m = X^{a_1} Y^{a_2} Z^{a_4} (X + Y + Z)^{a_3}.$$

This means the normalization of the cyclic cover of $\mathbb{P}^2$; if $\sum a_i = dm$, the coordinate W has weight d in $\mathbb{P}(1, 1, 1, d)$. The identification holds for every allowed t , not merely at a special fibre.

Domination and known algebraic cycles. Choose $\eta^m = -1$. The Fermat surface $F_m^2 : A^m + B^m + C^m + D^m = 0$ dominates $R_{m,a}$ by

$$[A : B : C : D] \longmapsto [A^m : B^m : C^m : \eta^{a_3} A^{a_1} B^{a_2} C^{a_4} D^{a_3}].$$

The Fermat curve $F_m : a^m + b^m + c^m = 0$ in turn gives a dominant rational map $F_m \times F_m \dashrightarrow F_m^2$ by

$$([a : b : c], [d : e : f]) \longmapsto [af : bf : \eta cd : \eta ce].$$

Both maps are generically finite: the first is finite over a general point of the plane, and the two independent ratios $a/b, d/e$ show that the second has two-dimensional image. Resolving the composition gives a generically finite morphism $\widetilde{F_m \times F_m} \rightarrow R_t$, with source obtained by point blowups.

The characters of $(\mu_m)^2$ in $H^1(F_m, \mathbb{C})$ occur once and are (a, b) with $a, b, a + b \neq 0$ in $\mathbb{Z}/m\mathbb{Z}$. Their rational Galois orbits have at most $\varphi(m) = 4$ elements. The rational group-algebra idempotents therefore decompose $J(F_m)$, up to isogeny, into abelian varieties of dimension at most two. By Theorem P.15(3), the Hodge conjecture holds for $J(F_m)^2 \times P$. The Abel–Jacobi maps give a surjection on cohomology to $F_m \times F_m \times P$: they are isomorphisms in degree one, and a polarization restricts to a nonzero class in degree two. This surjection splits in rational polarizable Hodge structures, so every Hodge class on the latter product lifts to a Hodge class, and is

algebraic. Point blowups of $F_m \times F_m$ add, after product with P , only Tate twists of $H^\bullet(P)$. The blowup formula and generically finite push–pull now prove the Hodge conjecture for $R_t \times P$.

An algebraic retraction, not just a Hodge isomorphism. Choose an Abel–Jacobi map followed by a σ -equivariant homomorphism $J(C) \rightarrow B$ that realizes the primitive factor, and let $a_B : C \times C \rightarrow B$ be its sum on the two coordinates. On T_C , the pullback a_B^* is the injective antisymmetrization

$$\bigwedge^2 V_j \longrightarrow V_j \otimes V_j, \quad v \wedge w \longmapsto v \otimes w - w \otimes v.$$

The same-factor $H^2(C)$ terms vanish since $2j \not\equiv 0 \pmod{m}$. The map r acts trivially on this tensor, while s acts by minus tensor interchange on $H^1(C) \otimes H^1(C)$ and therefore also fixes it. Quotient pushforward, composed with a_B^* , gives an algebraic correspondence inducing an injection $\alpha : T \hookrightarrow H^2(R_t, \mathbf{Q})$. For precision, resolve the quotient map equivariantly. The transfer identity is $q^*q_* = \sum_g g^*$ modulo exceptional Tate summands, so is $2m$ times the identity on a_B^*T ; the Tate corrections vanish because T has no Tate subquotient.

There is an algebraic projector p_T onto T in $H^2(B)$. To see this, choose a sufficiently general $k \in K$: the four eigenvalues $\tau(k)^2$ can be separated from all mixed products $\tau(k)\tau'(k)$, $\tau \neq \tau'$. Both sets are Galois stable, so a rational polynomial in k^* gives the projector. Clearing denominators realizes k by a rational algebraic correspondence; the cohomological-degree projector is algebraic on an abelian variety. Choose a K -compatible polarization, for example the one inherited from the Jacobian. The norm-one elements of K preserve its polarization form. Their characters on T are $\pm 2e_1, \pm 2e_2$, distinct from the mixed-summand characters $0, \pm e_1 \pm e_2$; hence T and its complement are orthogonal. This argument remains valid at CM and nonsimple parameters.

Let L_B be an ample Lefschetz operator. On primitive degree two, the polarized adjoint of α is

$$\alpha^\dagger = p_T L_B^{-2} \alpha : H^2(R_t, \mathbf{Q}) \longrightarrow T.$$

Here $L_B^{-2} : H^6(B) \rightarrow H^2(B)$ is algebraic by the Lefschetz standard conjecture for abelian varieties; see [8, §5]. The image of T is primitive on R_t , since its pairing with any ample class is a map to a Tate structure and hence zero. Positivity of these polarizations and injectivity of α make $M = \alpha^\dagger \alpha|_T$ invertible. Its inverse is a rational polynomial in M by Cayley–Hamilton. Consequently $\beta = M^{-1} \alpha^\dagger$ is an algebraic correspondence with $\beta \alpha = \text{id}_T$. For a Hodge class $h \in T \otimes H^2(P)$, the class $(\alpha \otimes 1)h$ is algebraic on $R_t \times P$ by the preceding paragraph. Applying $\beta \otimes 1$ proves the assertion. $\square$

Theorem 13.13 (A complete cyclic-five-cover family in N2). *Let $B_t = J(C_{5,t})$ with $\text{End}^0(B_t) = \mathbf{Q}(\zeta_5)$, and let $S = J(v^2 = z^5 - 1)$. Then the Hodge conjecture holds for $B_t \times S$. Its Hodge dimensions are $(1, 4, 12, 14, 12, 4, 1)$, including four exceptional degree-four classes. The endomorphism hypothesis holds for very general t , so this is a nonconstant family of sixfolds with a simple fourfold and a simple CM surface.*

Proof. The surface has CM by $K = \mathbf{Q}(\zeta_5)$ with type $\{1, 2\}$ and $\text{Hg}(S) = U_K$. Put $\gamma(\zeta_5) = \zeta_5^2$ and $\nu(s) = s\gamma(s)$ on U_K . This is an isogeny of rational tori. The actual joint group is

$$\{(g, s) \in U_K(H^1(B_t), h) \times U_K : \det_K g = \nu(s)\}.$$

Indeed the determinant Hodge cocharacter on B_t is $(2, 0)$, whereas on S it is $(1, 1)$; ν sends the latter to the former. The displayed rational group is connected, has the same dimension as $\text{Hg}(B_t)$, and contains the Hodge circle. Projection of the actual group onto $\text{Hg}(B_t)$ is surjective, proving equality. On embeddings 1, 2 the map ν is $(u, v) \mapsto (uv, v/u)$. Using the conjugate

of the second B_t block gives exactly the representation of Lemma 13.10. The four remaining classes are algebraic by Lemma 13.12; divisors and these classes generate the Hodge ring.

For nonemptiness, the branch data $(1, 1, 1, 2)$ are equivalent to case 11 of [5, Table 1]. This family fills its one-dimensional PEL component. Its generic group is the full unitary group just used, so its generic endomorphism algebra is K . This statement does not assert that every rational polarization form with this CM field occurs in the family. $\square$

Theorem 13.14 (All weighted quartic-cover families). *For any branch data in Lemma 13.12, put $B_t = B_{m,a,t}$ and choose*

$$P = \begin{cases} J(v^2 = z^5 - 1), & m = 5, 10, \\ E_i \times E_{\sqrt{-2}}, & m = 8, \\ E_i \times E_{\sqrt{-3}}, & m = 12, \end{cases}$$

where $E_{\sqrt{-d}}$ has CM by $\mathbf{Q}(\sqrt{-d})$ and E_i by $\mathbf{Q}(i)$. Then the Hodge conjecture holds for $B_t \times P$ for every $t \neq 0, 1$. Whenever B_t is simple, it holds for every power of $B_t \times P$. Very general parameters have simple B_t with $\text{End}^0(B_t) = K$.

Proof. The determinant cocharacter has one pure and one balanced block. For $m = 5, 10$, $K = \mathbf{Q}(\zeta_5)$ and the CM types of P are conjugate under field automorphisms. Orient the action at the pure block; the isogeny $s \mapsto s\gamma(s)$ of Theorem 13.13, with a suitable generator γ , sends its cocharacter to the fourfold determinant cocharacter. For $m = 8, 12$, orient both imaginary quadratic actions at that block and use $(u, v) \mapsto uv$ in U_K . Thus for non-CM B_t the joint group is precisely the connected mixed model of Lemma 13.10, by the same containment, connectedness, dimension, and surjectivity argument used there.

At a simple CM parameter the group is U_L , with L octic: for biquadratic K this follows from Proposition 14.6; for $K = \mathbf{Q}(\zeta_5)$ use the rank-three exclusion in the proof of Corollary 14.9. That exclusion uses only $K \subset L$ and simplicity, not the branch vector or covering degree. The joint group is the corresponding CM mixed model, whose generators are again divisors and the same Q . Lemma 13.12 makes Q algebraic at every parameter, and Proposition 13.11 proves the all-powers claim for simple B_t in both cases.

If B_t is not simple, each simple factor has dimension at most three. After adjoining P , the resulting sixfold belongs to the already proved rows N5–N10. This proves the sixfold assertion at these parameters; no all-powers assertion for these additional factorizations is needed here. Finally the cyclic-cover monodromy theorem [5, Proposition 4.7(i)] gives $\text{SU}(1, 1)$ on the balanced primitive block. The pure block excludes a diagonal identification of the two derived factors, and the determinant cocharacter spans U_K . The generic commutant is K ; hence the very general primitive factor is simple with precisely that endomorphism algebra. $\square$

Lemma 13.15 (Specializing a fixed Hodge subvariation over a curve). *Let $f : \mathcal{X} \rightarrow C$ be a smooth projective family over a smooth connected projective complex curve, and let $\mathcal{V} \subset R^{2p}f_*\mathbf{Q}$ be a rational polarizable subvariation everywhere of type (p, p) . If every class of $\mathcal{V}_c$ is algebraic for very general c , the same holds at every $c \in C$.*

Proof. After a finite étale cover, $\mathcal{V}$ is constant. Indeed its polarization, after the Tate twist, is positive definite and preserved by monodromy; intersection with an integral cohomology lattice gives a monodromy-stable lattice. Its orthogonal automorphism group is finite. Choose a very general base point outside the images of all nondominating irreducible components of all relative Chow schemes. There are only countably many such components, and each image is closed because the fixed-degree relative Chow schemes are projective. Represent a rational basis of $\mathcal{V}$ there by rational linear combinations of effective cycles. Every component containing

one of these effective cycles dominates C . By projective hyperplane sections through the chosen point, choose a curve in that component which still dominates C , and normalize it. Pulling back the universal cycle gives a cycle family over a finite cover of C ; its cohomology class is a flat section. Taking a common cover for the finitely many effective cycles in a basis, their indicated rational combinations extend to all fibers. On this connected cover the corresponding flat classes are the given constant basis of $\mathcal{V}$, since they agree at the chosen point. Restricting at a point over any desired c proves the assertion, and descent is harmless for rational cycles over C . Only classes in the specified subvariation are specialized; this does not assert that all new Hodge classes in a special fiber extend. $\square$

Theorem 13.16 (Completion of the represented one-dimensional PEL types). *Fix the marked rational polarized PEL datum of any family in Theorem 13.14, including its signatures and the matching fixed surface P . Every simple fourfold B with this datum satisfies the Hodge conjecture on all powers of $B \times P$. Likewise, every simple sixfold with the marked rational PEL datum of Theorem 10.5 satisfies the Hodge conjecture on all powers. Here equality of data means a K -linear identification of the rational Hermitian spaces up to compatible positive conformal scaling, in the same period-domain component. No new rational Hermitian form is added by this completion step.*

Proof. In each case H^1 has K -rank two and exactly one balanced embedding; the other embeddings are pure. Hence the PEL domain has dimension one. The K/K^+ Hermitian form is definite at a real place of K^+ , so it is anisotropic over K^+ . Its special unitary group has no nontrivial K^+ -rational unipotent elements: the image at that definite place is compact, where a unipotent unitary matrix must be the identity. The restriction-of-scalars derived group is semisimple and has no nontrivial rational unipotents. The arithmetic compactness criterion [4, Theorem 3.3(b)] makes each corresponding connected PEL curve compact. At a sufficiently fine level it is a smooth projective curve S carrying its universal abelian scheme.

After a finite cover of the original parameter curve, its primitive Jacobian factor has level structure and gives an algebraic morphism to S . The monodromy calculation used in the covering theorems shows that this morphism is nonconstant. It extends to the smooth projective completion C of the parameter curve because S is projective. The resulting nonconstant morphism $C \rightarrow S$ is surjective. Pull back the universal abelian scheme to C ; this is a smooth family also at points missing from the original covering parameter space. Changing a lattice only changes these realizations up to isogeny, so it suffices to treat this one fine-level component of the rational datum.

For the quartic datum, on the family times the fixed P , the space Q is a rank-four rational subvariation of R^4 , everywhere of type $(2, 2)$. This follows directly from the four determinant eigenspaces: their Hodge types and those of the fixed CM factor do not change with the period. It is polarizable as a subvariation of the smooth projective family. Lemma 13.12 makes Q algebraic over the original parameter curve, so Lemma 13.15 makes it algebraic at every point of C . At any simple fourfold point, the group dichotomy and the CM calculation in Theorem 13.14 still apply: their inputs are the field action and signatures, not the branch equation. Divisors and Q generate its sixfold Hodge ring, and Proposition 13.11 gives all powers.

For the sextic datum use the rank-two subvariation W_k of R^6 , everywhere of type $(3, 3)$. Theorem 10.5 proves its algebraicity on the covering parameter curve, and the same specialization lemma treats its completion. If a simple fiber is CM, Theorem 11.7 applies. Otherwise the rank-two-block calculation in Theorem 10.5, determined by signatures $(2, 0), (1, 1), (0, 2)$, gives the full derived group and norm-kernel determinant image; Proposition 9.3 then gives all powers from the algebraic Weil plane. This proves both assertions. The argument specializes cycles from a dense set in a curve to its missing points; it never deforms a cycle from a proper lower-dimensional subfamily into a higher-dimensional PEL space. $\square$

Example 13.17 (Two additional marked rational Hermitian types). The branch vectors

$$(m, \mathbf{a}) = (10, (1, 2, 2, 5)), \quad (12, (1, 1, 4, 6))$$

give the curves $y^{10} = x(x-1)^2(x-t)^2$ and $y^{12} = x(x-1)(x-t)^4$. Their genera are six and seven, respectively, by $g = 1 + m - \frac{1}{2} \sum_i \gcd(m, a_i)$. Their primitive factors both have dimension four. The preceding theorem proves the claimed N2 and N3 products at every parameter, and all powers when that factor is simple. At non-CM parameters their Hodge polynomial is $1 + 4t + 12t^2 + 14t^3 + 12t^4 + 4t^5 + t^6$.

These extend the rational Hermitian types in the earlier examples. For $x = \zeta_m^{a_1}, y = \zeta_m^{a_2}, z = \zeta_m^{a_3}$, the geometric skew-Hermitian intersection matrix of [10, Proposition 7.4] has determinant

$$D(\mathbf{a}) = \frac{1 - xyz}{(1-x)(1-y)(1-z)}.$$

Use the same imaginary scalar to turn each compared pair of skew forms into Hermitian forms, so this scalar cancels in their ratio. For the first comparison identify $\zeta_5 = \zeta_{10}^2$; for the second write $\sqrt{3} = 2\zeta_{12} - \zeta_{12}^3$. Exact calculation gives

$$\frac{D_{10}(1, 2, 2, 5)}{D_5(1, 1, 1, 2)} = 2, \quad \frac{D_{12}(1, 1, 4, 6)}{D_{12}(1, 1, 1, 9)} = 1 - \frac{\sqrt{3}}{3}.$$

At the prime over 2 of $\mathbf{Q}(\sqrt{5})$, the extension $\mathbf{Q}(\zeta_5)/\mathbf{Q}(\sqrt{5})$ is unramified quadratic and inert: 2 has order four modulo five, and its residue degree in the real subfield is two. The first ratio has valuation one. At $\pi = \sqrt{3}$ in $\mathbf{Q}(\sqrt{3})$, the extension obtained by adjoining i is unramified and inert, since $X^2 + 1$ is irreducible over $\mathbf{F}_3$. The second ratio equals $(\pi - 1)/\pi$ and has valuation -1 . Local norms in an unramified quadratic extension have even valuation, so neither ratio is a relative norm. A basis change multiplies a Hermitian determinant by a relative norm; conformal scaling in rank two multiplies it by a square, also a norm. Thus each new marked rational type differs from the indicated old one. Theorem 13.16 covers all simple members of each represented marked rational PEL type, not only the explicitly parametrized fibers. This is a comparison within this manuscript, not a claim of priority or of distinctness of every branch orbit.

Remark 13.18 (Finite branch catalogue and its precise scope). Modulo permutations of the four branch points and multiplication of all exponents by a unit modulo m , the hypotheses of Lemma 13.12 give the following representatives:

m	Branch representatives
5	(1, 1, 1, 2)
8	(1, 1, 1, 5), (1, 1, 2, 4), (1, 2, 2, 3)
10	(1, 1, 2, 6), (1, 1, 3, 5), (1, 2, 2, 5), (1, 2, 3, 4)
12	(1, 1, 1, 9), (1, 1, 2, 8), (1, 1, 3, 7), (1, 1, 4, 6), (1, 2, 3, 6), (1, 2, 4, 5), (1, 3, 4, 4), (1, 3, 10, 10), (2, 3, 3, 4)

The enumeration ranges over $1 \leq a_1, a_2, a_3 < m$, sets $a_4 = -a_1 - a_2 - a_3 \pmod{m}$, and applies exactly the nonzero, gcd, and multiplicity tests in the lemma. Canonicalizing by all units and sorting gives 1, 3, 4, 9 orbits, from 16, 112, 240, 496 ordered types. The script `verify_weighted_quartic.py` independently checks unordered quadruples and reconstructs all ordered orbits, and verifies the two determinant ratios. All seventeen branch families are covered by Theorem 13.14; they are not claimed to represent seventeen distinct rational PEL types. Exhaustion of these finite covering data does not exhaust all forms in N2 or N3.

Determinant stabilization and the residual N2 cycle

We use one additional external assertion, denoted (Aux): the auxiliary-spin proposition in the section *Degree-four tensors in spin families* of [19]. For a totally real Galois field $F/\mathbf{Q}$ and an F -quadratic space (V', q') of dimension $d' \geq 5$, $d' \equiv 5 \pmod{8}$, signature $(2, d' - 2)$ at every real place and Witt index at least two, choose a positive period plane at each real place. The complete spin Hodge structure

$$W' = \text{Res}_{F/\mathbf{Q}} C_F^+(V') = H^1(A', \mathbf{Q})$$

then has the following specified algebraic tensor. If z' is a volume element, sum

$$\mathcal{P}_\sigma = \sum_j L_{v_j z'} \otimes L_{v_j^\vee z'} \quad \text{over } \sigma : F \hookrightarrow \mathbf{C}, \quad (\text{N2.1})$$

where the bases are q' -dual and each operator is supported on the σ summand. After polarization identifies the two endomorphism factors with four weight-one slots, the resulting rational tensor $\mathcal{P}$ is the class of a cycle in $\text{CH}^2((A')^4)_{\mathbf{Q}}$. The assertion holds for every choice of period planes and every isogenous realization. We use precisely this assertion as an external hypothesis; it is stronger than the specified K3 Kuga–Satake correspondence alone. Its proof in the new preprint is not independently verified here.

Theorem 13.19 (Residual N2 from the auxiliary-spin tensor). *Assume (Aux) and the CM Hodge theorem [15, Theorem 1.1]. For every coupled non-CM case $A = B \times P$ of Proposition 13.6, there is $Y \in \text{CH}^2(A)_{\mathbf{Q}}$ with nonzero projection to its mixed space Q . Consequently the rational Hodge conjecture holds for every A^n , $n \geq 1$. There is no restriction on the primitive quartic field or on the rational Hermitian form.*

Proof. *Quadratic spaces with complementary period planes.* Put $V = H^1(B, \mathbf{Q})$, $K = \text{End}^0(B)$ and $F = K^+$; this F is the real quadratic subfield, not the CM endomorphism field of P . Denote the pure and mixed real places of F by c, a . Choose a rank-one K -CM weight-one structure M with CM type $\{\bar{\sigma}_c, \sigma_a\}$, realized by a CM abelian surface C_M . Set

$$U = V \otimes_K M, \quad R = \bigwedge_K^2 V.$$

Both are weight-two structures, of F -ranks four and two. The first has rational Hodge numbers $(1, 6, 1)$; its period plane is at a , whereas that of R is at c . Compatible Hermitian polarizations and an F -scalar with prescribed real signs give invariant F -quadratic forms with signatures

$$\text{sig}(U_c) = (0, 4), \quad \text{sig}(U_a) = (2, 2), \quad \text{sig}(R_c) = (2, 0), \quad \text{sig}(R_a) = (0, 2). \quad (\text{N2.2})$$

The positive planes are exactly the indicated period planes. Here the norm form on the K -line R may be scaled independently at the two real places. All these forms are invariant under $G = \text{Hg}(B \times C_M)$.

Let $N_7 = \langle -1, \dots, -1 \rangle_F$ have rank seven and trivial G -action, and put $V' = U \perp R \perp N_7$. Its F -dimension is thirteen and both signatures are $(2, 11)$. At finite places a quadratic form of dimension at least five is isotropic. Hasse–Minkowski therefore splits one hyperbolic plane over F ; its complement has dimension eleven and signatures $(1, 10)$, so a second application splits another. Thus the Witt index is at least two. Since $F/\mathbf{Q}$ is Galois, all hypotheses of (Aux) hold.

A rational spin lift with the original central character. Fix a complex embedding of F . Let V_σ^+ be the two-dimensional block at one extension to K , put $d = \det V_\sigma^+$, and let m denote the corresponding character of M . A maximal isotropic subspace of $(U \perp R)_\sigma$ is

$$L = (V_\sigma^+ \otimes m) \oplus \mathbf{C}_d, \quad \det L = d^2 m^2.$$

The inverse image of $\mathrm{GL}(L)$ in Spin_6 is $\{(g, t) : t^2 = \det g\}$. Taking the actual character $t = dm$ gives a lift of $G_{\overline{\mathbf{Q}}}$ and its Fock module $S = (dm)^{-1} \wedge^\bullet L$. On a maximal torus, write b_1, b_2 for the weights of V_σ^+ , so $d = b_1 b_2$. The complete character calculation is

Fock states	weights
1	$(dm)^{-1}$
e_1, e_2	b_2^{-1}, b_1^{-1}
e_3	m^{-1}
e_{12}	m
e_{13}, e_{23}	b_1, b_2
e_{123}	dm .

(N2.3)

The two-dimensional blocks are representations of the full group, not just identities of torus weights: $V_\sigma^+ \otimes d^{-1} \simeq (V_\sigma^+)^V$ and the e_{13}, e_{23} block is V_σ^+ itself. In particular the center is correct.

The G -action on V' is rational. Two lifts through the finite central kernel of $\mathrm{Res}_{F/\mathbf{Q}} \mathrm{Spin}(V') \rightarrow \mathrm{Res}_{F/\mathbf{Q}} \mathrm{SO}(V')$ agree, since G is geometrically connected. The lifts just constructed at the two embeddings, extended trivially across N_7 , therefore descend uniquely to $\mathbf{Q}$. Their Hodge circle is the spin lift of the chosen period planes. Explicitly, at c the exponents of d, m are 2, -1 , and at a they are 0, 1; every weight in (N2.3) has exponent 1 or -1 . Thus these are the actual weight-one constituents of W' .

Over $\mathbf{C}$, restriction of the Spin_{13} spinor to $\mathrm{Spin}_6 \times \mathrm{Spin}_7$ is the full eight-dimensional Spin_6 spinor tensored with the eight-dimensional Spin_7 spinor. The full even Clifford regular module contains this with nonzero multiplicity. Formula (N2.3) at both F -embeddings consequently contains all of $V_{\mathbf{C}}$. Base change for rational G -Hom spaces and semisimplicity show that V is a rational summand of W' . The subspace $(W')^{G^{\mathrm{der}}}$ is a rational weight-one Hodge summand with torus Hodge group; denote its CM abelian variety by D . Thus B and D are isogeny factors of A' . The factor maps and polarization adjoints are algebraic abelian homomorphisms. No individual complex half-spinor is asserted to have a rational model of the same dimension.

The nonzero mixed block. In (N2.3), use e_1, e_2 for the isotropic U directions and e_3 for the R direction. Up to a nonzero normalization, the U -part of (N2.1) acts on the Spin_6 factor as

$$C_U = \sum_{i=1}^2 (\varepsilon_i \otimes \iota_i + \iota_i \otimes \varepsilon_i),$$

where ε_i is exterior multiplication and ι_i contraction. Its CM-output projection is

$$C_U(e_{13} \otimes e_{23}) = e_3 \otimes e_{123} - e_{123} \otimes e_3, \quad C_U(e_{13} \wedge e_{23}) = 2e_3 \wedge e_{123} \neq 0. \quad (\text{N2.4})$$

The R operators have zero projection to these CM states. Operators from N_7 preserve the B/CM decomposition of the Spin_6 factor, through its parity operator, and also give zero. In odd dimension the volume element acts by a nonzero scalar on the spinor, so $L_{vz'}$ has the same nonzero block as Clifford multiplication by v . Multiplicity factors can be selected by weight-one intertwiners. Hence no other summand cancels (N2.4).

By (Aux), rational weight-one factor maps applied to its cycle, followed by the two diagonals, yield a rational cycle $Z \in \mathrm{CH}^2(B \times D)_{\mathbf{Q}}$ whose component

$$0 \neq z \in T_B \otimes H^2(D, \mathbf{Q}), \quad T_B = \bigwedge_K^2 V, \quad (\text{N2.5})$$

is nonzero. To justify rationality of this assertion, the complex selectors in (N2.4) lie in the complexifications of the rational intertwiner spaces. If every tuple of rational selectors had zero determinant projection after the diagonals, multilinearity would force every complex tuple to

have zero projection. Formula (N2.4) contradicts this. Polarization converts the input determinant line to its dual line, which is still a summand of T_B ; the nonzero projection is unaffected.

Transport to the given CM surface. The four determinant characters of T_B form a single Galois orbit and are distinct for the full unitary group. Thus T_B is a simple rational Hodge structure. The tensor (N2.5) defines an injection $T_B^\vee(-2) \rightarrow H^2(D, \mathbf{Q})$, with image C . The given coupled space $Q \neq 0$ similarly supplies an isomorphic summand $C_P \subset H^2(P, \mathbf{Q})$. Choose a Hodge projection onto C and an isomorphism $C \rightarrow C_P$; their composite is a rational Hodge map $\phi : H^2(D, \mathbf{Q}) \rightarrow H^2(P, \mathbf{Q})$. Writing $d_D = \dim D$, its Poincare-dual class lies in

$$H^{2d_D-2}(D, \mathbf{Q}) \otimes H^2(P, \mathbf{Q}). \quad (\text{N2.6})$$

Since $D \times P$ is CM, the external CM theorem makes this precise rational Hodge class algebraic: choose $\Gamma \in \text{CH}^{d_D}(D \times P)_{\mathbf{Q}}$ representing it. Then

$$Y = (\text{pr}_{BP})_*(\text{pr}_{BD}^* Z \cdot \text{pr}_{DP}^* \Gamma) \in \text{CH}^2(B \times P)_{\mathbf{Q}}.$$

Its codimension is $2 + d_D - d_D = 2$ and its mixed component is $(1 \otimes \phi)z \neq 0$. The single-cycle criterion of Proposition 13.6 and Proposition 13.11 finish the proof. $\square$

Example 13.20. Choose $(B_1, \dots, B_2)$ jointly very generally in $\prod_i \mathcal{A}_{d_i}(N)$ with $d_i = (4, 2)$, as constructed in Proposition P.7. Then each factor is simple with $\text{End}^0(B_i) = \mathbf{Q}$, and the factors are pairwise nonisogenous. Their product is an actual sixfold of this partition, with $\text{End}^0(A) = \mathbf{Q}^2$ and the proved joint group above. The Hodge-space dimensions in codimensions 0 through 6 are $(1, 2, 3, 3, 3, 2, 1)$. An explicit monomial basis over $\mathbf{Q}$ on A is $\theta_1^{a_1} \cdots \theta_2^{a_2}$ with $0 \leq a_i \leq d_i$; every such monomial is an intersection of divisors. On powers add Poincare divisors between the copies of each individual B_j . There are no cross-factor divisors or higher mixed invariants in this verified jointly generic example.

14 N3: The $4 + 1 + 1$ case (Proved)

Scope and status. The independent cases, the imaginary-quadratic endomorphism stratum, and every repeated elliptic pair are proved. The new external CM theorem also settles all remaining cases with a CM fourfold. Theorem 14.11 covers the residual non-CM fourfold with biquadratic CM endomorphism field and two distinct matching CM elliptic fields, using the specified complete KS correspondence. It supplies the required nonzero mixed projection for every rational Hermitian type and proves all powers of each residual model.

Corollary 14.1 (All remaining CM-fourfold cases in N3). *Assume the CM input in Theorem P.9. If B, E_1, E_2 are CM, then the Hodge conjecture holds on all powers of $B \times E_1 \times E_2$, whether or not the elliptic fields coincide or the joint torus splits as a product. Together with Proposition 14.5, this settles every sixfold in N3 whose fourfold factor is CM.*

Proof. The all-CM product and each of its powers are covered by Lemma P.12. Proposition 14.5 leaves only two nonisogenous CM elliptic factors as a possible residual case. Hence, when B is CM, the preceding argument settles every case that proposition leaves open. $\square$

Example 14.2 (Distinct elliptic fields with a CM fourfold). Take the simple CM fourfold B of Example 13.2, and elliptic curves E_1, E_2 with CM by $k_1 = \mathbf{Q}(\sqrt{-3})$ and $k_2 = \mathbf{Q}(\sqrt{-15})$, respectively. Both fields are contained in $\mathbf{Q}(\zeta_{15})$ and are distinct. For the type $\{1, 2, 4, 7\}$, the multiplicities for the four embeddings of $k_1 k_2$, indexed by signs, are $(n_{++}, n_{+-}, n_{-+}, n_{--}) =$

$(2, 1, 1, 0)$ after orientation. Thus the fourfold has the pure-mixed quartic pattern rather than two balanced quartic signatures. Nevertheless all powers of $B \times E_1 \times E_2$ satisfy the Hodge conjecture by Corollary 14.1, including their mixed classes.

Proposition 14.3. *Let $A \sim B_1 \times \cdots \times B_3$ with dimensions $(4, 1, 1)$. Assume the actual joint Hodge group is the independent product of the full symplectic groups on these factors. Then all Hodge classes on all powers of A are divisor-generated, and $P_A(t) = (1 + t + \cdots + t^4)(1 + t)(1 + t)$.*

Proof. Use Lemma P.1 to replace the isogenous variety by the product. Under the stated group equality,

$$\left(\bigwedge \bigoplus_i H^1(B_i)^{\oplus n} \right)^{\text{Hg}(A)} = \bigotimes_i \left(\bigwedge H^1(B_i)^{\oplus n} \right)^{\text{Sp}(H^1(B_i))}.$$

Each factor is generated by degree-two contractions by Lemma P.2. These are divisor classes on the product, so every invariant is algebraic. On one copy the generators are the polarization powers through the dimension of each factor, giving the displayed polynomial. $\square$

Proposition 14.4 (Quadratic-center and independent strata). *Let $A = B \times E_1 \times E_2$, with B a simple fourfold. The Hodge conjecture holds if no CM field of an E_i embeds in the center of $\text{End}^0(B)$. It also holds whenever $\text{End}^0(B)$ itself is an imaginary quadratic field k , including both E_i with CM by k .*

Proof. Collect repeated elliptic isogeny classes. In the first case apply Theorem P.15(2) successively to the distinct classes: their Hodge groups separate from B , and distinct elliptic classes separate from one another. The fourfold theorem and divisor generation on elliptic products then prove algebraicity.

Suppose $\text{End}^0(B) = k$. Over $\mathbb{C}$ write $H^1(B) = W \oplus W^*$ with $\dim W = 4$. The two constituents are irreducible and nonisomorphic. The faithful irreducible minuscule representations of dimension four are the standard SL_4 , standard Sp_4 , and $2 \otimes 2$ of $\text{SL}_2 \times \text{SL}_2$. In the latter two cases the representation is self-dual. For a nontrivial derived Hodge circle its weights are symmetric; translating them to the two weights $1, -1$ forces central weight zero and multiplicities $(2, 2)$. A trivial derived circle would force the entire group to be a torus and is impossible here. Thus in these self-dual cases the determinant circle is trivial; all its conjugates are trivial, the connected center vanishes, and $W \cong W^*$ contradicts the commutant. Hence the derived group is SL_4 . Pure signatures are excluded by simplicity. For signature $(2, 2)$ the group is SU_4 ; for $(3, 1)$ or $(1, 3)$ it is the full unitary group, with determinant-circle exponent ± 2 .

In the balanced case SU_4 has no elliptic quotient, so all elliptic factors separate. In the unbalanced case only elliptic curves with CM by k can interact. Let their multiplicity be $m \leq 2$, and orient their k -action opposite to the larger signature on B . The joint group is the connected kernel of $\det(v)u^2 = 1$ in $U(W) \times U_k$: it contains the Hodge circle and has the same dimension as $U(W)$, onto which the actual group projects. The remaining elliptic factors separate. On the single B factor, SL_4 -invariants are pairings and the degree-four volumes, whose central weights are $\det^{\pm 1}$. To cancel a volume requires elliptic weight ∓ 2 . This is impossible for $m < 2$ and, for $m = 2$, uses exactly two same-sign elliptic coordinates. The only new classes are therefore the two conjugate lines of $W_k(B \times E^2)$ in degree six. This product has signature $(3, 3)$ and a rank-one k -elliptic summand, so Lemma P.16 makes its Weil plane algebraic. All other invariants are divisor products. $\square$

Proposition 14.5 (All repeated elliptic factors). *For every simple fourfold B and elliptic curve E , the Hodge conjecture holds for $B \times E^2$, without a restriction on $\text{End}^0(B)$. More generally, for $B \times E_1 \times E_2$ the only cases not covered here have E_1, E_2 nonisogenous CM curves with both CM fields embedding in the center of $\text{End}^0(B)$.*

Proof. Moonen–Zarhin’s fivefold classification [3, Theorem (0.2) and §5.11] shows that $\text{Hg}(B \times E)$ is the independent product unless E has CM by k acting on B with signature $(3, 1)$, up to interchange. Repeating E does not change this group. In the independent case the Hodge ring on $B \times E^2$ is a tensor product, and its classes are algebraic by the fourfold theorem and elliptic divisor generation.

In the exceptional case the same classification gives a CM field $F = \text{End}^0(B)$ containing k , the full unitary Lefschetz group $G = U_F(H^1(B), h)$, and the connected joint group

$$H = \{(g, u) \in G \times U_k : \det_k(g)u^2 = 1\}.$$

Here the action on E is opposite to the larger k -signature of B . Write $[F : \mathbf{Q}] = 2e$ and $q = 4/e$, so $e = 1, 2, 4$ and $G_{\mathbf{C}} = \prod_{j=1}^e \text{GL}(W_j)$ with $\dim W_j = q$. Orient the embeddings over k consistently. The kernel of $\det_k$ has derived factors $\text{SL}(W_j)$ and determinant relation $\prod_j d_j = 1$. Its exterior invariants on the single B factor are pairings and the two simultaneous volumes, of degree $eq = 4$. Indeed central neutrality forces all signed determinant exponents to agree; on one copy they are 0, 1, or -1 .

The volumes acquire elliptic weights ∓ 2 . In the exterior algebra of two copies of $H^1(E)$, weight 2 or -2 consists of the unique product of the two same-sign coordinates. Hence the only additional invariants are the two lines of $W_k(B \times E^2)$ in degree six; the other invariants are divisor products. The rank-one elliptic summand allows Lemma P.16, proving their algebraicity.

Finally, if the two elliptic curves are nonisogenous and one has no CM field embedding in the center of $\text{End}^0(B)$ (including a non-CM curve), Proposition (3.8) of [3] separates that curve from B times the other curve. In the CM case its field also differs from that of the other elliptic curve. The remaining fivefold has algebraic Hodge classes by Theorem (0.2) of [3] and the fourfold theorem. Their exterior products prove the claim. $\square$

Proposition 14.6 (The full residual structure in N3). *Let $A = B \times E_1 \times E_2$ be an N3 case not already covered by Proposition 14.5. Write $k_i = \text{End}^0(E_i)$. Then $k_1 \neq k_2$ are imaginary quadratic fields in the center of $\text{End}^0(B)$, and $K = k_1 k_2$ acts on $H^1(B)$ with signatures $(2, 0), (1, 1)$, after orienting the two actions. Put*

$$T = \bigwedge_K^2 H^1(B, \mathbf{Q}), \quad Q = (T \otimes H^1(E_1, \mathbf{Q}) \otimes H^1(E_2, \mathbf{Q})) \cap H^{2,2}(A).$$

Here T has the same summand interpretation as in Lemma 13.12. There are exactly two alternatives:

1. $\text{End}^0(B) = K$, $\text{Hg}(B)_{\mathbf{C}} = \text{GL}_2^2$, and $P_A(t) = 1 + 4t + 12t^2 + 14t^3 + 12t^4 + 4t^5 + t^6$;
2. B is simple CM with field $L \supset K$ of degree eight, $\text{Hg}(B) = U_L$, and

$$P_A(t) = (1 + t)^6 + 4t^2(1 + t)^2 = 1 + 6t + 19t^2 + 28t^3 + 19t^4 + 6t^5 + t^6.$$

In both cases the actual joint group is the connected group

$$H = \{(g, u, v) \in \text{Hg}(B) \times U_{k_1} \times U_{k_2} : \det_K g = uv \text{ in } U_K\}.$$

The rational space Q has dimension four and, together with divisors, generates the entire Hodge ring of A .

Proof. The field and embedding restrictions follow from Proposition 14.5. Since the center contains K , the dimension formula for the simple algebra $\text{End}^0(B)$ forces it to be a CM field

of degree four or eight. Index the four embeddings of K by their signs on k_1, k_2 . Their $H^{1,0}$ multiplicities have the form

$$(n_{++}, n_{+-}, n_{-+}, n_{--}) = (a, b, 2 - b, 2 - a), \quad 0 \leq a, b \leq 2.$$

The signatures of k_1, k_2 are given by $a + b$ and $a + 2 - b$. Neither can be 0 or 4: a pure imaginary-quadratic action would make B isogenous to a fourth power of an elliptic curve. If $a = b = 1$, both K determinants are trivial on the Hodge circle and hence on its rational Hodge group. That group lies in SL_2^2 , so each standard block is isomorphic to its dual. The resulting off-diagonal intertwiners in the complex commutant do not commute with $K \otimes \mathbf{C}$, contradicting centrality of K in $\mathrm{End}^0(B)$. Enumerating the remaining pairs (a, b) , and conjugating the two actions if necessary, gives $(a, b) = (2, 1)$. In particular both k_i have signature $(3, 1)$.

If the endomorphism field is K , the derived-factor and determinant argument in Lemma 13.12 gives the full unitary group. If it is L of degree eight, choose the four embeddings of L over the positive embedding of k_1 . The Hodge cocharacter has three entries 1 and one entry -1 . Galois elements fixing k_1 act transitively on these four embeddings. Their orbit contains all four such vectors, the rows of $J_4 - 2I_4$, whose determinant is -16 . Thus the Hodge group has rank four and is the full torus U_L .

Orient the elliptic actions at the pure K embedding. The map $\nu : U_{k_1} \times U_{k_2} \rightarrow U_K$, $(u, v) \mapsto uv$, is an isogeny with complex coordinates $(uv, u/v)$. It sends the elliptic Hodge cocharacter $(1, 1)$ to $(2, 0)$, the determinant cocharacter of B . Hence the displayed rational group contains the joint Hodge circle. The map $\det_K$ has connected kernel: over $\mathbf{C}$ it is either the two ordinary determinants on GL_2^2 , or $(a_1, a_2, b_1, b_2) \mapsto (a_1 a_2, b_1 b_2)$ on $\mathbf{G}_m^4$. Thus H is connected. Its dimension equals $\dim \mathrm{Hg}(B)$, and the actual joint group projects onto $\mathrm{Hg}(B)$, proving equality. Using only the two squared norm relations would leave an extraneous connected component; the primitive relation $\det_K g = uv$ specifies the actual group.

The non-CM invariant calculation is Lemma 13.10. In the CM case, each two-dimensional block splits into two torus characters. Its determinant-kernel invariants are the two individual pairings, their product, and the two volumes. Thus its weight-zero polynomial is $(1 + t)^2$, while its volume weights and degrees are unchanged. The same neutrality equations again allow only one active volume block. The divisor polynomial is $(1 + t)^6$ and the additional contribution is $4t^2(1 + t)^2$. These four degree-four lines descend to $\mathbf{Q}$; multiplication by divisors generates all remaining degrees. $\square$

Corollary 14.7 (A single-cycle criterion for all residual N3 cases). *In Proposition 14.6, the Hodge conjecture holds for A if and only if there is a cycle $Z \in \mathrm{CH}^2(A)_{\mathbf{Q}}$ whose cohomology class has nonzero projection to Q .*

Proof. The projector onto T is a rational polynomial in a general K endomorphism on $H^2(B)$, as in Lemma 13.12; it is algebraic. Tensoring it with the degree-one projectors of the two elliptic curves gives an algebraic projector which sends a Hodge class to its Q component. The field $k_1 \otimes_{\mathbf{Q}} k_2 = K$ acts on $H^1(E_1) \otimes H^1(E_2)$ by algebraic correspondences. Consequently Q is a K -module. Its rational dimension is four, so it is one-dimensional over K ; every nonzero vector generates it. A cycle with nonzero projection therefore makes all of Q algebraic. Proposition 14.6 proves sufficiency, and necessity is immediate. This is a criterion for each fixed A ; it does not assert existence or deformability of the required cycle. $\square$

Theorem 14.8 (Two coupled cyclic-cover families in N3). *For $m = 8$ or 12 , let $B_{m,t}$ be as in Lemma 13.12, with $\mathrm{End}^0(B_{m,t}) = \mathbf{Q}(\zeta_m)$. Let E_1 have CM by $k_1 = \mathbf{Q}(i)$, and let E_2 have CM by*

$$k_2 = \mathbf{Q}(\sqrt{-2}) \quad (m = 8), \quad k_2 = \mathbf{Q}(\sqrt{-3}) \quad (m = 12).$$

Then the Hodge conjecture holds for $B_{m,t} \times E_1 \times E_2$, with Hodge dimensions $(1, 4, 12, 14, 12, 4, 1)$. The endomorphism hypothesis holds for very general t in each family.

Proof. Here $K = k_1 k_2$. Orient both elliptic actions positively at the pure-signature embedding of $B_{m,t}$. The rational torus isogeny

$$\nu : U_{k_1} \times U_{k_2} \longrightarrow U_K, \quad (u, v) \longmapsto uv$$

has complex coordinates $(uv, u/v)$ after choosing the other embedding over the same embedding of k_1 . The determinant Hodge cocharacter of the fourfold is $(2, 0)$, the image under ν of the elliptic cocharacter $(1, 1)$. Since the two elliptic curves are nonisogenous, their joint group is $U_{k_1} \times U_{k_2}$. As in Theorem 13.13, connectedness and dimension give the actual joint group as $\det_K(g) = \nu(u, v)$. Lemma 13.10 identifies all extra classes in $T \otimes H^2(E_1 \times E_2)$, and Lemma 13.12 algebraizes them.

Finally, the primitive eigenspace multiplicities computed in that lemma are $(2, 1, 1, 0)$. The cyclic-cover monodromy result [5, Proposition 4.7(i)] gives full $SU(1, 1)$ projection on the balanced block. Rational Galois conjugation gives both complex SL_2 projections. They cannot be diagonally identified, since one real unitary factor is compact and the other is not. The determinant cocharacter spans both central directions. Thus the generic Hodge group is full GL_2^2 after complexification and its commutant is precisely K . The primitive factor, not the whole Jacobian, is the fourfold here. $\square$

Corollary 14.9 (All parameters in the three cyclic-cover families). *The Hodge conjecture holds for all $t \in \mathbf{C} \setminus \{0, 1\}$ in the products of Theorems 13.13 and 14.8, without the assumption $\text{End}^0(B_{m,t}) = \mathbf{Q}(\zeta_m)$. If the fourfold factor remains simple but becomes CM, the Hodge dimensions are $(1, 6, 19, 28, 19, 6, 1)$ instead of $(1, 4, 12, 14, 12, 4, 1)$.*

Proof. If $B_{m,t}$ is not simple, every simple factor has dimension at most three. Its product with the specified surface or elliptic pair belongs to the already proved rows N5–N10. Assume it is simple. The dichotomy in Lemma 13.12 shows that it either has the stated quartic endomorphism field or is CM with an octic field L . The former case was proved above. For $m = 8, 12$, Proposition 14.6 supplies the CM generators, and the now unrestricted Lemma 13.12 algebraizes Q at this parameter.

For $m = 5$, write $K = \mathbf{Q}(\zeta_5) \subset L$. We first prove $\text{Hg}(B_{5,t}) = U_L$. The eight distinct CM weights of a simple fourfold force its Hodge rank to be at least three: a rank- r orbit basis of signed Hodge cocharacters gives at most 2^r distinct weight columns. If its rank were three, [3, Lemma (3.7)] would give an imaginary quadratic field $k \subset L$ with $\text{Hg}(B_{5,t}) = \ker N_{L/k}$ and k -signature $(2, 2)$. Since K has no imaginary quadratic subfield, $L = Kk$ is Galois with group $C_4 \times C_2$. Let γ generate $\text{Gal}(L/k)$, and let τ fix K and conjugate k . Complex conjugation is $c = \gamma^2 \tau$. A balanced CM type, indexed over k by $1, \gamma, \gamma^2, \gamma^3$, has two plus and two minus signs. If the plus signs are adjacent, γ^2 negates the sign vector and τ fixes the CM type; if they are opposite, γ^2 itself fixes the type. In either case the type is induced from a proper subfield, contrary to simplicity. Rank three is impossible.

Thus $\text{Hg}(B_{5,t}) = U_L$. Replacing GL_2^2 in the group of Theorem 13.13 by its maximal torus U_L gives the actual joint group: $\det_K = N_{L/K}$ has connected kernel, and the same connectedness, Hodge-circle and dimension argument applies. Its weights are those in the CM calculation of Proposition 14.6. Divisors and the same four-dimensional $Q \subset T \otimes H^2(S)$ generate its Hodge ring, and Lemma 13.12 proves their algebraicity. This argument treats each special fibre directly; it does not infer algebraicity by an unsupported deformation or specialization step. $\square$

The complete Kuga–Satake construction for residual N3

We use the following explicit external input. For a projective K3 surface X , let C_X be the *complete* even-Clifford Kuga–Satake abelian variety of $T(X)$. The specified standard map

$$\kappa_X : T(X) \longrightarrow H^1(C_X, \mathbf{Q})^{\otimes 2}$$

is induced by a rational cycle in $\mathrm{CH}^2(X \times C_X^2)_{\mathbf{Q}}$. This is the assertion of [17, Theorem 1.1], used as an external hypothesis, not independently proved here. We only need it for the Picard-number-fourteen surfaces constructed below.

Lemma 14.10 (Connected Spin descent). *Let G be a geometrically connected algebraic group over a field F of characteristic zero and $r : G \rightarrow \mathrm{SO}(W, q)$ an F -morphism. If $r_{\overline{F}}$ lifts to $\mathrm{Spin}(W, q)$, then that lift is unique and descends to F .*

Proof. Two lifts differ by a morphism from the connected group $G_{\overline{F}}$ to μ_2 , necessarily the identity character. Every Galois conjugate of a lift is therefore the same lift, proving descent. $\square$

Theorem 14.11 (Residual N3 under the specified KS correspondence). *Assume the external Kuga–Satake assertion above. Let $A = B \times E_1 \times E_2$ be in the non-CM alternative of Proposition 14.6: B is simple, $\mathrm{End}^0(B) = K = k_1 k_2$ for distinct imaginary quadratic fields, E_i has CM by k_i , and the K -multiplicities are $(2, 1, 1, 0)$. Then there is a cycle $Z \in \mathrm{CH}^2(A)_{\mathbf{Q}}$ with nonzero projection to the four-dimensional space Q of that proposition. In particular the rational Hodge conjecture holds for every A^n , $n \geq 1$. No Hermitian discriminant or covering-family restriction is imposed.*

Proof. A geometric K3-type structure. Write $k_1 = \mathbf{Q}(a)$, $a^2 = -d$ with $d > 0$, and $V = H^1(B, \mathbf{Q})$. The rational Hodge projector

$$p = \frac{1}{2}(1 + (a_B \otimes a_{E_1})/d) \quad \text{on } V \otimes H^1(E_1, \mathbf{Q})$$

has an eight-dimensional image W . Orient the elliptic characters u, v as in the proof of Proposition 14.6, and write $V_C = V_1 \oplus V_1^\vee \oplus V_2 \oplus V_2^\vee$, where $\det V_1 = uv$ and $\det V_2 = u/v$. Here $\vee$ denotes the dual Hodge-group representation; the Hodge weights are assigned separately. Then

$$W_C = V_1 u^{-1} \oplus V_2 u^{-1} \oplus V_1^\vee u \oplus V_2^\vee u, \quad h(W) = (1, 6, 1).$$

The derived group SL_2^2 has no fixed vector in W , so W has no rational $(1, 1)$ class. The restricted geometric polarization gives a rational quadratic form q of signature $(2, 6)$, using the K3 cup-product sign. Scale it to an even integral lattice Λ_W . Its discriminant group has length at most eight. The inequalities $2 < 3$, $6 < 19$ and $\ell(A_{\Lambda_W}) + 2 \leq 10 < 22 - 8$ give a primitive embedding in the K3 lattice by Nikulin’s criterion [18, Theorem 14.1.12]. Period-map surjectivity [18, Theorem 6.3.1] then gives a projective K3 surface X with $\rho(X) = 14$ and a rational polarized Hodge isometry

$$T(X) \simeq (W, q). \tag{N3.1}$$

The isometry (N3.1) is not assumed algebraic.

The rational Spin lift and KS factors. Over $\mathbf{C}$, set $L_i = V_i u^{-1}$ and $L = L_1 \oplus L_2$. The quadratic pairing identifies $W_C = L \oplus L^\vee$, and

$$\det L = (\det V_1)(\det V_2)u^{-4} = u^{-2}.$$

For the standard inclusion $\mathrm{GL}(L) \subset \mathrm{SO}(L \oplus L^\vee)$, its Spin inverse image is $\{(g, t) : t^2 = \det g\}$, acting on $\wedge^\bullet L$ by $t^{-1} \wedge^\bullet g$. Taking $t = u^{-1}$ gives a lift of the actual connected joint Hodge group of Proposition 14.6 over $\overline{\mathbf{Q}}$. The action on (W, q) is rational, so Lemma 14.10 gives a rational

group lift $H \rightarrow \text{Spin}(W, q)$. Its Hodge circle is the KS lift of the period circle; the induced action on the complete even Clifford module $U = H^1(C_X, \mathbf{Q}) = \text{Cl}^+(W, q)$ is its actual weight-one Hodge structure.

The two complex half-spin representations are

$$S^+ = u \bigwedge^{\text{even}} L = u \oplus v \oplus v^{-1} \oplus u^{-1} \oplus (V_1 \otimes V_2 \otimes u^{-1}), \quad S^- = u \bigwedge^{\text{odd}} L = V_1 \oplus V_2 \oplus V_1^\vee \oplus V_2^\vee.$$

The equalities are of $H_{\mathbf{C}}$ -modules; they do not posit rational models of individual half-spinors. The *complete* regular module satisfies

$$U_{\mathbf{C}} \simeq (S^+)^{\oplus 8} \oplus (S^-)^{\oplus 8}.$$

Since H is reductive, scalar extension of rational H -Hom spaces to $\mathbf{C}$ identifies them with their complex counterparts. Thus $V, H^1(E_1), H^1(E_2)$ are rational Hodge summands of U ; indeed one obtains an isogeny

$$C_X \sim B^8 \times E_1^8 \times E_2^8 \times D, \quad \dim D = 16. \quad (\text{N3.2})$$

Only the first three types of factors are needed. Their rational weight-one inclusions, projections and polarization adjoints are induced by rational abelian homomorphisms.

Algebraic extraction of the needed matrix coefficients. Choose $t_0 \in W_{\mathbf{Q}}$ with $q(t_0) \neq 0$. In the endomorphism realization, the specified KS map is

$$\kappa_0 : W(1) \longrightarrow \text{End}(U), \quad \kappa_0(w)(x) = wxt_0; \quad (\text{N3.3})$$

see [18, Chapter 4, Proposition 2.6 and Remark 2.7]. The polarization of U converts (N3.3) into the tensor map in the external KS hypothesis. Right multiplication $R_r : x \mapsto xr$ by $r \in \text{Cl}^+(W, q)$ commutes with the left KS Hodge action and is therefore an algebraic rational endomorphism of C_X . Postcomposing (N3.3) with R_r replaces t_0 by $t_0 r$. Since t_0 is invertible, $t_0 \text{Cl}^+ = \text{Cl}^-$: every odd right matrix coefficient can be selected by algebraic right multiplication.

Over $\mathbf{C}$ write

$$\text{Cl}^+ = \text{End}(S^+) \oplus \text{End}(S^-), \quad \text{Cl}^- = \text{Hom}(S^-, S^+) \oplus \text{Hom}(S^+, S^-).$$

The right matrix index is a trivial H -multiplicity space. If e_1, e_2 is a basis of L_1 , the vectors $1, e_1 \wedge e_2 \in S^+$ have characters u, v , respectively. In the regular summand $S^+ \otimes (S^+)^{\vee}$, choose a multiplicity covector $\lambda \in (S^+)^{\vee}$. In $S^- \otimes (S^-)^{\vee}$, choose a right coefficient $\tau \in S^-$ and an odd matrix $b : S^- \rightarrow S^+$ with $\lambda(b\tau) = 1$; such b is among the complex linear combinations of $t_0 r$ above. Evaluate the output at τ and project its left index to $V_1 \subset S^-$. The resulting complex matrix components of (N3.3) are

$$\alpha_1(w) = \text{pr}_{V_1} c(w)1, \quad \alpha_2(w) = \text{pr}_{V_1} c(w)(e_1 \wedge e_2), \quad (\text{N3.4})$$

viewed respectively in $H^1(B) \otimes H^1(E_1)$ and $H^1(B) \otimes H^1(E_2)$ by elliptic polarization and the corresponding Tate twist. The weights on both sides are two. These component selectors are complex H -equivariant maps on weight-one factors of U . Rational H -Hom spaces base-change to the complex Hom spaces; hence (N3.4) belongs to the complex span of maps induced by finitely many rational algebraic correspondences

$$\mathcal{G}_i \subset \text{CH}^2(X \times B \times E_i)_{\mathbf{Q}}. \quad (\text{N3.5})$$

Concretely, apply the rational factor maps in (N3.2), their polarization adjoints, and right multiplication to the two abelian slots of the KS cycle. No algebraicity of the abstract embedding $W \subset H^2(B \times E_1)$ is used.

Nonzero contraction and rational cycle. Let $f_1, f_2 \in L_1^\vee$ be dual to e_1, e_2 . In the exterior Clifford model $c(e_i) = e_i \wedge (-)$ and $c(f_i) = \iota_i$. If b_1, b_2 denote the corresponding basis of V_1 , then

$$\frac{\alpha_1}{\left| \begin{array}{cc} & e_1 \quad e_2 \\ b_1 u^{-1} & b_2 u^{-1} \end{array} \right|}, \quad \frac{\alpha_2}{\left| \begin{array}{cc} & f_1 \quad f_2 \\ b_2 v^{-1} & -b_1 v^{-1} \end{array} \right|}.$$

The opposite entries and the L_2 terms vanish after projection to V_1 . Contracting with q^{-1} and pulling the two B slots back along the diagonal gives a nonzero scalar multiple of

$$(b_1 \wedge b_2) \otimes u^{-1} \otimes v^{-1}. \quad (\text{N3.6})$$

Indeed the $L_1 \otimes L_1^\vee$ terms give $b_1 \otimes b_2 - b_2 \otimes b_1$; the normalization of q changes only the nonzero scalar. Since $\det V_1 = uv$, (N3.6) is an H -invariant in the target $\mathcal{Q}_C$.

To realize this contraction by rational cycles, choose a rational point $x_0 \in X$ and a basis D_a of $\text{NS}(X)_{\mathbf{Q}}$, with $G_{ab} = D_a \cdot D_b$. The algebraic transcendental projector

$$\pi_X^{\text{tr}} = \Delta_X - x_0 \times X - X \times x_0 - \sum_{a,b} (G^{-1})_{ab} D_a \times D_b \in \text{CH}^2(X \times X)_{\mathbf{Q}} \quad (\text{N3.7})$$

has cohomology class equal to the inverse pairing on $T(X)$. For $\Gamma_i \in \mathcal{G}_i$, form their exterior product, intersect it with the pullback of π_X^{tr} on the two X slots, push forward along $X \times X$, and pull back by

$$\delta : A \longrightarrow (B \times E_1) \times (B \times E_2), \quad (b, e_1, e_2) \longmapsto ((b, e_1), (b, e_2)).$$

Call the resulting rational cycle $F(\Gamma_1, \Gamma_2)$. Its codimension is $4 + 2 - 4 = 2$ before δ^* and remains two after it. By (N3.6), the complexification of this bilinear construction has a pair of inputs with nonzero Q -projection. If every pair of rational correspondences in (N3.5) had zero projection, all complex linear combinations would too, a contradiction. Thus some $Z = F(\Gamma_1, \Gamma_2) \in \text{CH}^2(A)_{\mathbf{Q}}$ has nonzero Q -projection. Corollary 14.7 makes all of Q algebraic; Proposition 13.11 gives the assertion for all powers. $\square$

Remark 14.12 (Logical scope of Theorem 14.11). The proof uses the *specified* full even-Clifford KS map as an external algebraicity input. It does not turn a complex half-spinor into a rational abelian factor, assume the Hodge isometry (N3.1) algebraic, or claim that an individual complex line in (N3.6) is a rational cycle. The rational descent occurs in the full regular module and the finite rational correspondence spaces (N3.5).

Example 14.13. Choose $(B_1, \dots, B_3)$ jointly very generally in $\prod_i \mathcal{A}_{d_i}(N)$ with $d_i = (4, 1, 1)$, as constructed in Proposition P.7. Then each factor is simple with $\text{End}^0(B_i) = \mathbf{Q}$, and the factors are pairwise nonisogenous. Their product is an actual sixfold of this partition, with $\text{End}^0(A) = \mathbf{Q}^3$ and the proved joint group above. The Hodge-space dimensions in codimensions 0 through 6 are $(1, 3, 4, 4, 4, 3, 1)$. An explicit monomial basis over $\mathbf{Q}$ on A is $\theta_1^{a_1} \cdots \theta_3^{a_3}$ with $0 \leq a_i \leq d_i$; every such monomial is an intersection of divisors. On powers add Poincare divisors between the copies of each individual B_i . There are no cross-factor divisors or higher mixed invariants in this verified jointly generic example.

Example 14.14 (A mixed Weil plane requiring both elliptic factors). Take a very general simple $k = \mathbf{Q}(i)$ fourfold with Hermitian matrix $\text{diag}(1, 1, 1, -1)$, and let $E : y^2 = x^3 - x$ have the opposite k -action. The PEL argument gives $\text{Hg}(B) = U(3, 1)$ and $\text{End}^0(B) = k$. On $A = B \times E^2$ the total signature is $(3, 3)$ and the determinant is -1 . Proposition 14.4 yields

$$P_A(t) = (1 + t + \cdots + t^4)(1 + 4t + t^2) + 2t^3, \quad (1, 5, 6, 8, 6, 5, 1).$$

The five divisor classes consist of one on B and four on E^2 . They generate the ordinary part; the two additional middle classes are the algebraic Weil plane. With only one elliptic factor the weight 2 of the fourfold volume could not be cancelled.

15 N4: The 3 + 3 case (Proved)

Scope and status. Proved for all simple threefold pairs, including nonisogenous unitary and CM factors. The actual Hodge-circle calculation below rules out extra central couplings; the remaining Weil plane is algebraic.

Proposition 15.1. *Let $A \sim B_1 \times \cdots \times B_2$ with dimensions $(3,3)$. Assume the actual joint Hodge group is the independent product of the full symplectic groups on these factors. Then all Hodge classes on all powers of A are divisor-generated, and $P_A(t) = (1 + t + \cdots + t^3)(1 + t + \cdots + t^3)$.*

Proof. Use Lemma P.1 to replace the isogenous variety by the product. Under the stated group equality,

$$\left(\bigwedge \bigoplus_i H^1(B_i)^{\oplus n} \right)^{\text{Hg}(A)} = \bigotimes_i \left(\bigwedge H^1(B_i)^{\oplus n} \right)^{\text{Sp}(H^1(B_i))}.$$

Each factor is generated by degree-two contractions by Lemma P.2. These are divisor classes on the product, so every invariant is algebraic. On one copy the generators are the polarization powers through the dimension of each factor, giving the displayed polynomial. $\square$

Lemma 15.2 (Two sextic CM factors: the actual Hodge-circle test). *Let B, C be nonisogenous simple CM threefolds with endomorphism fields F_B, F_C . Their joint Hodge group is $U_{F_B} \times U_{F_C}$ unless the fields contain a common imaginary quadratic field k . In that case orient their k -signatures oppositely; the joint group is*

$$\ker(N_{F_B/k} N_{F_C/k} : U_{F_B} \times U_{F_C} \longrightarrow U_k).$$

Proof. We work with rational cocharacter modules, on which the absolute Galois group acts through a finite quotient. The joint Hodge torus is determined by the rational span of the Galois orbit of its Hodge cocharacter. For either sextic field this cocharacter has coordinates $f \in \{1, -1\}^3$ on three conjugate embedding pairs, and its orbit spans the full three-dimensional module, by [3, §2.3]. All finite-group modules below are semisimple over $\mathbf{Q}$. Thus a failure of a product must identify isomorphic irreducible constituents, and the corresponding projections of the two Hodge cocharacters must satisfy the proposed graph relation.

First suppose a sextic CM field F contains no imaginary quadratic subfield. Its module is irreducible by [3, Proposition (2.4)]. Let G be its faithful signed permutation group and $P \subset S_3$ its transitive permutation quotient, so $P = C_3$ or S_3 . The sign subgroup $N = G \cap \{\pm 1\}^3$ contains complex conjugation $c = (-1, -1, -1)$. As an $\mathbf{F}_2[P]$ -submodule it is either $\langle c \rangle$ or all of $\mathbf{F}_2^3$: the permutation module is the direct sum of the constant line and the irreducible sum-zero plane for C_3 . If $N = \langle c \rangle$, the product of the three signs is a character $\epsilon : G \rightarrow \{\pm 1\}$ with $\epsilon(c) = -1$. It is trivial on the stabilizer of a positive embedding coordinate. To check this, that stabilizer projects either to the identity or to a transposition fixing that coordinate. In the latter case a negative product of the other two signs would square to a nonconstant sign vector, contradicting $N = \langle c \rangle$. The quadratic field cut out by ϵ would therefore lie in F and be imaginary, a contradiction. Hence $N = \{\pm 1\}^3$ and $G = N \rtimes P$.

The subgroup N is characteristic: it is the largest normal 2-subgroup, since P has none. Its three eigenlines in $\mathbf{Q}^3$ are precisely the coordinate lines. If two such irreducible modules were identified, their faithful Galois quotients would agree, and an intertwiner would permute these lines. Transitivity forces its three nonzero coefficients to have the same absolute value. Compatibility with the two vectors in $\{\pm 1\}^3$ forces that value to be one. Consequently it identifies the actual weight sets $\{\pm e_1, \pm e_2, \pm e_3\}$ of $H^1(B)$ and $H^1(C)$ and their Hodge circles. Their complex

representations on the joint torus would then have nonzero intertwiners. Formation of this intertwiner space commutes with extension from $\mathbf{Q}$ to $\mathbf{C}$, yielding a nonzero rational Hodge morphism $H^1(B) \rightarrow H^1(C)$. It would give an isogeny, a contradiction. There is therefore no common quotient in this case. Nor can this irreducible three-dimensional module share a quotient with the $1 + 2$ decomposition considered next.

Now let $F = kF_0$, with F_0 real cubic. The field k is unique, since two imaginary quadratic subfields would yield a real quadratic subfield of F_0 . If Ω is the normal closure of F_0 , then $\text{Gal}(k\Omega/\mathbf{Q}) = C_2 \times P$, where $P = C_3$ or S_3 . Write $V = \{(x_1, x_2, x_3) : \sum x_i = 0\}$ for the standard rational plane, and χ_k for the quadratic character. The cocharacter module is

$$\chi_k \oplus (\chi_k \otimes V).$$

The first constituent can be shared only when the quadratic fields agree. A simple CM type is not constant, since a constant type is induced from k . Reverse the k -orientation to arrange $\sum f_i = 1$. Its projection to V is then $\frac{2}{3}v_j$, where $v_1 = (-2, 1, 1)$ and v_2, v_3 are its coordinate permutations.

The two-dimensional constituent is faithful on $C_2 \times P$. For $P = C_3$ its endomorphism field is cyclic sextic; all nonconstant CM types are conjugate under its field automorphisms, up to complex conjugation. Two factors sharing this constituent would have the same normal closure and hence the same field, and would be isogenous. For $P = S_3$, equality of faithful constituents gives the same normal closure L and the same maximal real subfield $\Omega = L^{(c)}$. The real cubic subfields of Ω are conjugate, so identify their permutation modules. There are exactly two imaginary quadratic subfields of L , with characters differing by sgn_P . If the two k 's coincide, any intertwiner on V is scalar because $\text{End}_{\mathbf{Q}[P]}(V) = \mathbf{Q}$. Two vectors among v_1, v_2, v_3 are proportional only when equal. A compatible graph therefore identifies the CM types under a field isomorphism and makes the factors isogenous.

If the two k 's differ, an intertwiner is, up to a nonzero rational scalar, $J = \rho - \rho^{-1}$, with $\rho = (123)$ on V . Indeed J commutes with ρ and anticommutes with each transposition, so it intertwines V and $V \otimes \text{sgn}_P$; these irreducible modules have one-dimensional intertwiner space. Each Jv_j has one zero coordinate, whereas every v_i has no zero coordinate. Hence no such intertwiner carries one projected Hodge cocharacter to the other. The orbit spans both copies, and there is no two-dimensional coupling. This check is necessary even though the abstract quotient modules are isomorphic.

Only the one-dimensional constituent for a common k can remain coupled. Its Hodge-circle exponents are 1 and -1 after opposite orientation. Thus its relation is precisely the displayed product of norms. Over $\mathbf{C}$ this is a primitive relation on six torus coordinates, so its kernel is connected. The orbit-span calculation gives equality of Lie algebras, hence equality of the connected groups. $\square$

Proposition 15.3 (Threefold strata). *Let B, C be simple threefolds. The Hodge conjecture holds for $B \times C$ in each of the following cases: B, C are isogenous; at least one has Type I; or one has imaginary quadratic endomorphism algebra and the other is CM. It also holds if their joint group is a product, or is the full determinant kernel*

$$\{(g_1, g_2) \in U_k(B) \times U_k(C) : \det_k(g_1) \det_k(g_2) = 1\},$$

where their k -signatures have been oriented as $(2, 1)$ and $(1, 2)$. The last displayed group equality is a hypothesis of this intermediate proposition; Theorem 15.4 below removes the need to assume it in the classification of the whole row.

Proof. For isogenous factors the prime-dimensional theorem applies to B^2 . For nonisogenous factors with a Type I factor, [3, Proposition (2.4), Lemma (3.4)] gives the product group; its invariants are products of divisor classes.

Suppose $\text{End}^0(B) = k$ is imaginary quadratic and $\text{End}^0(C) = F$ is sextic CM. Their individual groups are $U_k(B)$ and U_F . Any common quotient must factor through $\det_k : U_k(B) \rightarrow U_k$. A rank-one quotient of U_F is isogenous to $U_{k'}$ for an imaginary quadratic subfield $k' \subset F$, by [3, Lemma (3.7)] applied to its connected kernel. Nontrivial coupling therefore requires $k \subset F$. If this fails, the group is a product. If $k \subset F$, both determinant Hodge circles have exponents ± 1 : a pure k -signature on C would contradict simplicity. After reversing one action the joint group equals

$$H = \{(g, t) : \det_k(g) N_{F/k}(t) = 1\}.$$

Indeed it contains the Hodge circle. Its kernel over U_F is SU_3 , which also lies in the actual joint derived group, since a torus has trivial derived group and projection onto SU_3 is surjective. Consequently both groups have dimension $3 + 8 = 11$ and coincide. The kernel is connected, as is seen from the primitive determinant relation over C .

For invariants, first take SL_3 on the B coordinates and $\ker N_{F/k}$ on the C coordinates. In each single factor the only invariants besides pairings are its degree-three volumes, by the calculation in Lemma P.19. Central neutrality allows the product of one volume from each factor, with matching determinant weights. These two complex lines form $W_k(B \times C)$; all other invariants are divisor products. Its k -rank-three summand B is odd, so Lemma P.16 proves algebraicity. For the displayed determinant-kernel hypothesis the same calculation uses $\text{SL}_3 \times \text{SL}_3$ instead and gives the identical conclusion. $\square$

Theorem 15.4 (The complete $3 + 3$ row). *For any two simple abelian threefolds B, C , all Hodge classes on $B \times C$ are algebraic. If they are nonisogenous, their joint group is a product except possibly for the single determinant relation associated with a common imaginary quadratic field. In that exceptional case the Hodge ring on the sixfold is generated by divisors and $W_k(B \times C)$.*

Proof. Proposition 15.3 treats isogenous factors, Type I factors, and a unitary factor paired with a CM factor. Lemma 15.2 treats two CM factors. It remains to justify the group assertion for two nonisogenous non-CM factors with imaginary quadratic endomorphism fields k_B, k_C .

Their individual groups are U_3 and their derived groups are $\mathbf{Q}$ -simple forms of SL_3 , by [3, §2.3]. The quadratic field recovered from the Galois action on the two vertices of their A_2 Dynkin diagram is k_B , respectively k_C . If these fields differ, the derived groups have no common simple quotient, and their determinant tori have distinct quadratic Galois characters. Goursat's lemma gives the product group.

Suppose $k_B = k_C = k$ and orient both signatures as $(2, 1)$ for this paragraph. The joint group lies in $\det_k g_B = \det_k g_C$ because the two determinant Hodge circles have exponent one. A failure of independence of the derived groups would identify their SL_3 factors. On the chosen k -eigenspaces their standard three-dimensional modules would then be either equivalent or dual. The normalized Hodge circle has weights $(1, 1, -1)$; its traceless part has weights $(2/3, 2/3, -4/3)$. Duality negates this multiset and cannot preserve it. In the equivalent case the central scalar actions also agree, since the determinant relation makes their traces equal. Thus the two modules are equivalent for the full connected joint group. Complex conjugation gives the same conclusion for the other k -eigenspaces. As above, base change for Hom gives a nonzero rational Hodge morphism, contradicting nonisogeny. The derived group is consequently the full product. The determinant-circle image generates the diagonal norm-one torus. The actual group equals the connected determinant kernel by dimension; reversing the action on C writes its relation as $\det_k g_B \det_k g_C = 1$.

For the remaining CM–CM case, Lemma 15.2 gives the product or the same norm-determinant kernel. In a product group the prime-dimensional theorem makes every invariant divisor-generated. In a determinant kernel, first take the kernel of the individual determinant on each factor. Over $\mathbf{C}$ this is SL_3 in a unitary factor and $\{(z_1, z_2, z_3) : z_1 z_2 z_3 = 1\}$ in a CM factor. On one copy of either factor its exterior invariants are generated by degree-two pairings and the two degree-three volumes. In the CM case this follows because an exponent vector in $\{-1, 0, 1\}^3$ is trivial on the norm kernel only when its three coordinates agree; the unitary case is the standard SL_3 invariant calculation. The residual determinant torus permits only the opposite-weight product of a volume from each factor, besides products of pairings. The two resulting degree-six lines form the rational plane $W_k(B \times C)$. Its algebraicity follows from Lemma P.16, since each factor has k -rank three. All other classes are products of divisors. Rational descent is as in Lemma P.2; this proves the theorem on the entire sixfold. $\square$

Example 15.5. Choose $(B_1, \dots, B_2)$ jointly very generally in $\prod_i \mathcal{A}_{d_i}(N)$ with $d_i = (3, 3)$, as constructed in Proposition P.7. Then each factor is simple with $\mathrm{End}^0(B_i) = \mathbf{Q}$, and the factors are pairwise nonisogenous. Their product is an actual sixfold of this partition, with $\mathrm{End}^0(A) = \mathbf{Q}^2$ and the proved joint group above. The Hodge-space dimensions in codimensions 0 through 6 are $(1, 2, 3, 4, 3, 2, 1)$. An explicit monomial basis over $\mathbf{Q}$ on A is $\theta_1^{a_1} \cdots \theta_2^{a_2}$ with $0 \leq a_i \leq d_i$; every such monomial is an intersection of divisors. On powers add Poincaré divisors between the copies of each individual B_i . There are no cross-factor divisors or higher mixed invariants in this verified jointly generic example.

Example 15.6 (Two nonisogenous unitary threefolds). Over $k = \mathbf{Q}(i)$ use Hermitian matrices $h_B = \mathrm{diag}(1, 1, -1)$ and $h_C = \mathrm{diag}(1, -1, -1)$. Choose a jointly very general point of the product of their two-dimensional PEL domains. Derived monodromy is the independent product $\mathrm{SU}(h_B) \times \mathrm{SU}(h_C)$. The determinant circle has coordinates $(1, -1)$, so the joint group is exactly the determinant kernel in Proposition 15.3. Its commutant is $k \times k$, proving that the factors are simple and nonisogenous. The sixfold has split determinant -1 , and

$$P_A(t) = (1 + t + t^2 + t^3)^2 + 2t^3, \quad (1, 2, 3, 6, 3, 2, 1).$$

The two polarizations generate the ordinary part and $W_k(A)$ supplies the two algebraic middle classes. The group hypothesis has been verified for this family; it is not inferred merely from the partition $3 + 3$.

Example 15.7 (Two CM threefolds with a mixed Weil plane). Let $F_1 = \mathbf{Q}(\zeta_7 + \zeta_7^{-1})$, $F_2 = \mathbf{Q}(\zeta_9 + \zeta_9^{-1})$, and $K_j = \mathbf{Q}(i)F_j$. Each K_j is cyclic sextic. Choose signed CM types $(1, 1, -1)$ and $(-1, -1, 1)$ relative to $\mathbf{Q}(i)$, respectively, and let B, C be the corresponding CM threefolds. These nonconstant types are primitive: the only proper CM subfield is $\mathbf{Q}(i)$ and an induced type would be constant. Thus the factors are simple with endomorphism fields K_j . They are nonisogenous, since these fields are nonisomorphic (their ramification at 7 differs). Lemma 15.2 gives the joint rank-five norm kernel. The six opposite-coordinate pairings and the two mixed volumes give

$$P_{B \times C}(t) = (1 + t)^6 + 2t^3, \quad (1, 6, 15, 22, 15, 6, 1).$$

The twenty ordinary middle classes are divisor products; the two additional classes form the algebraic plane $W_{\mathbf{Q}(i)}(B \times C)$. This is a coupled CM example of the complete theorem, rather than an independence assumption.

16 N5: The $3 + 2 + 1$ case (Proved)

Scope and status. Proved for every sixfold in the $3 + 2 + 1$ row, including CM factors and the exceptional threefold–elliptic interaction.

Theorem 16.1. *Let $A \sim B \times S \times E$, where B, S are simple of dimensions three and two and E is elliptic. Then the Hodge conjecture holds for A . Its Hodge ring is generated by divisors and the pullback of the Weil plane on $B \times E$, when the matching imaginary-quadratic case occurs.*

Proof. Lemma P.21 separates $\text{Hg}(S)$ from $\text{Hg}(B \times E)$. The Kunneth invariant ring is therefore the tensor product of the two factor Hodge rings. The surface ring is divisor-generated by the prime-dimensional theorem. Lemma P.19 with $g = 3$ gives exactly the stated generators on the fourfold $B \times E$, and proves their algebraicity using the fourfold Weil theorem. Their exterior products with surface divisor classes are algebraic, proving the result. $\square$

Example 16.2. Choose $(B_1, \dots, B_3)$ jointly very generally in $\prod_i \mathcal{A}_{d_i}(N)$ with $d_i = (3, 2, 1)$, as constructed in Proposition P.7. Then each factor is simple with $\text{End}^0(B_i) = \mathbf{Q}$, and the factors are pairwise nonisogenous. Their product is an actual sixfold of this partition, with $\text{End}^0(A) = \mathbf{Q}^3$ and the proved joint group above. The Hodge-space dimensions in codimensions 0 through 6 are $(1, 3, 5, 6, 5, 3, 1)$. An explicit monomial basis over $\mathbf{Q}$ on A is $\theta_1^{a_1} \cdots \theta_3^{a_3}$ with $0 \leq a_i \leq d_i$; every such monomial is an intersection of divisors. On powers add Poincare divisors between the copies of each individual B_i . There are no cross-factor divisors or higher mixed invariants in this verified jointly generic example.

Example 16.3 (A fourfold Weil plane inside the $3 + 2 + 1$ row). Take a very general $k = \mathbf{Q}(i)$ threefold B with Hermitian datum $\text{diag}(1, 1, -1)$, the curve $E : y^2 = x^3 - x$ with opposite k -action, and a very general simple surface S with $\text{End}^0(S) = \mathbf{Q}$. Then $\text{Hg}(B) = U(2, 1)$, and Lemma P.21 separates S . The Hodge ring is generated by divisors and the pullback of the algebraic plane $W_k(B \times E)$. The dimension polynomial is

$$P_A(t) = (1 + 2t + 4t^2 + 2t^3 + t^4)(1 + t + t^2), \quad (1, 3, 7, 8, 7, 3, 1).$$

Thus this example includes exceptional classes and their products with divisors on the surface.

17 N6: The $3 + 1 + 1 + 1$ case (Proved)

Scope and status. Proved for every sixfold in the $3 + 1 + 1 + 1$ row. Elliptic factors may be isogenous or have CM.

Theorem 17.1. *If $A \sim B \times E_1 \times E_2 \times E_3$, where B is a simple threefold and the E_i are elliptic, the Hodge conjecture holds for A . The generators are divisors and pullbacks of fourfold Weil classes from B times a matching CM elliptic factor, as in Lemma P.20.*

Proof. Apply that lemma to $P = E_1 \times E_2 \times E_3$ and use isogeny invariance. Grouping repeated elliptic classes is included in its proof, so no independence assumption on the three elliptic curves is imposed. $\square$

Example 17.2. Choose $(B_1, \dots, B_4)$ jointly very generally in $\prod_i \mathcal{A}_{d_i}(N)$ with $d_i = (3, 1, 1, 1)$, as constructed in Proposition P.7. Then each factor is simple with $\text{End}^0(B_i) = \mathbf{Q}$, and the factors

are pairwise nonisogenous. Their product is an actual sixfold of this partition, with $\text{End}^0(A) = \mathbf{Q}^4$ and the proved joint group above. The Hodge-space dimensions in codimensions 0 through 6 are $(1, 4, 7, 8, 7, 4, 1)$. An explicit monomial basis over $\mathbf{Q}$ on A is $\theta_1^{a_1} \cdots \theta_4^{a_4}$ with $0 \leq a_i \leq d_i$; every such monomial is an intersection of divisors. On powers add Poincare divisors between the copies of each individual B_i . There are no cross-factor divisors or higher mixed invariants in this verified jointly generic example.

Example 17.3 (Repeated matching elliptic factors). Use the $k = \mathbf{Q}(i)$ threefold B and matching elliptic curve E from Example 16.3, and put $A = B \times E^3$. Pairings contribute $(1 + t + t^2 + t^3)(1 + 9t + 9t^2 + t^3)$. A threefold volume must be paired with elliptic weight ∓ 1 . Choosing a positive and $a + 1$ negative coordinates in the three elliptic copies gives $2 \binom{3}{a} \binom{3}{a+1}$ such invariants, for $a = 0, 1, 2$. Hence

$$P_A(t) = (1 + t + t^2 + t^3)(1 + 9t + 9t^2 + t^3) + 6t^2 + 18t^3 + 6t^4,$$

with dimensions $(1, 10, 25, 38, 25, 10, 1)$. Lemma P.20 supplies algebraic generators: divisors and pullbacks of $W_k(B \times E)$ under the k -linear maps $E^3 \rightarrow E$.

18 N7: The $2 + 2 + 2$ case (Proved)

Scope and status. The Hodge conjecture is proved for the full $2 + 2 + 2$ row by an established product-of-surfaces theorem. A uniform explicit list of all joint CM character relations is not claimed here.

Theorem 18.1. *Every $A \sim S_1 \times S_2 \times S_3$, with S_i simple abelian surfaces, satisfies the rational Hodge conjecture, as does every power.*

Proof. Ramón Marí's theorem in Theorem P.15(3) applies to the three abelian surfaces. A power of this product is again a product of abelian surfaces, so the same theorem applies. Finally use isogeny invariance. The theorem permits arbitrary CM fields, isogenies among factors, and common Hodge-group quotients; pairwise group independence is not used as a substitute for its hypothesis. $\square$

Example 18.2 (An explicit repeated CM surface). Let $S = J(C)$ for $C : y^2 = x^5 - 1$. The curve has genus two, and $(x, y) \mapsto (\zeta_5 x, y)$ gives CM by $F = \mathbf{Q}(\zeta_5)$ with type $\{1, 2\}$. Its signed Galois orbit contains $(1, 1)$ and $(1, -1)$, so $\text{Hg}(S) = U_F$ has rank two. The four distinct eigencharacters have commutant F , proving $\text{End}^0(S) = F$ and simplicity. For $A = S^3$, each of its two conjugate character pairs has multiplicity three. Neutral monomials choose the same number of each sign, so

$$P_A(t) = (1 + 9t + 9t^2 + t^3)^2, \quad (1, 18, 99, 164, 99, 18, 1).$$

They factor into degree-two pairings. Rational descent and Lefschetz $(1, 1)$ show that the eighteen rational divisor classes generate the Hodge ring, including the Poincare classes between repeated factors. Here $\text{End}^0(A) = M_3(F)$.

19 N8: The $2 + 2 + 1 + 1$ case (Proved)

Scope and status. Proved for every sixfold in the $2 + 2 + 1 + 1$ row, without restrictions on endomorphisms or repetition of factors.

Theorem 19.1. *Every $A \sim S_1 \times S_2 \times E_1 \times E_2$ in this row satisfies the rational Hodge conjecture, as does every power.*

Proof. Apply Theorem P.15(3), since all four factors have dimension at most two. This includes both isogenous and nonisogenous factors; A^n has the same dimension bound on its simple factors. Isogeny invariance transfers the assertion to A . $\square$

Example 19.2 (A surface square and an elliptic square). Let S be the explicit CM surface of Example 18.2 and $E : y^2 = x^3 - x$. The field $\mathbf{Q}(i)$ does not embed in $\mathbf{Q}(\zeta_5)$, whose only quadratic subfield is real. Thus $\text{Hg}(S \times E) = U_{\mathbf{Q}(\zeta_5)} \times U_{\mathbf{Q}(i)}$, by Theorem P.15(2). For $A = S^2 \times E^2$, the three independent conjugate character pairs each have multiplicity two, and

$$P_A(t) = (1 + 4t + t^2)^3, \quad (1, 12, 51, 88, 51, 12, 1).$$

Opposite-character pairings give all invariants and therefore twelve rational divisor generators. This verifies a repeated CM case directly, as well as by the general theorem.

20 N9: The $2 + 1 + 1 + 1 + 1$ case (Proved)

Scope and status. Proved for every sixfold in the $2 + 1 + 1 + 1 + 1$ row, including all CM and repeated-factor cases.

Theorem 20.1. *Every $A \sim S \times E_1 \times E_2 \times E_3 \times E_4$ in this row satisfies the rational Hodge conjecture, as does every power.*

Proof. All simple factors have dimension at most two, so apply Theorem P.15(3) and isogeny invariance, also to every power. Alternatively, the simple surface has no imaginary quadratic subfield in the center of its endomorphism algebra; [3, Proposition (3.8)] separates it from the elliptic isogeny classes. Divisor generation on surface powers and on elliptic products then gives the same conclusion. $\square$

Example 20.2 (One CM surface and four equal elliptic factors). Use S and E from the preceding example and put $A = S \times E^4$. Their independent Hodge groups give

$$P_A(t) = (1 + t)^2(1 + 16t + 36t^2 + 16t^3 + t^4), \quad (1, 18, 69, 104, 69, 18, 1).$$

Two divisors from S and sixteen from E^4 generate the Hodge ring; the latter include graph and Poincare classes associated with the $\mathbf{Q}(i)$ -linear homomorphisms between elliptic copies. All classes are algebraic intersections of these divisors.

21 N10: The $1 + 1 + 1 + 1 + 1 + 1$ case (Proved)

Scope and status. Full row: every product of six elliptic curves, with arbitrary CM behavior and isogenous repetitions, on every power.

Theorem 21.1. *If a sixfold A is isogenous to a product of six elliptic curves, every Hodge class on every power of A is generated by divisor classes.*

Proof. Reduce by isogeny and collect equal isogeny classes. On powers one has $A^n \sim \prod_i E_i^{m_i n}$ with the E_i pairwise nonisogenous. A non-CM elliptic curve has Hodge group SL_2 . A subdirect product coupling two non-CM SL_2 factors would be a twisted diagonal by Goursat's lemma. Automorphisms of $\mathfrak{sl}_2$ are inner, so their standard Hodge representations would be isomorphic. The corresponding Hodge isomorphism of H^1 comes from a rational homomorphism of elliptic curves and would make them isogenous. This is excluded. The non-CM joint group is therefore the full product of its SL_2 factors. For a CM elliptic curve with imaginary quadratic field K_i , its Hodge group is the one-dimensional norm-one torus of K_i . Distinct isogeny classes of CM elliptic curves have distinct imaginary quadratic fields. In the anti-conjugate character space the one-dimensional rational Galois modules for these tori are the distinct quadratic characters of these fields. Their direct sum is multiplicity-free. The signed Hodge cocharacter has a nonzero component in every summand. Applying the rational character projectors to its Galois orbit therefore spans every summand. The joint CM Hodge group is the product of these tori. Finally no connected common quotient of a semisimple product of SL_2 groups and a torus is nontrivial: a quotient of the former has semisimple Lie algebra and a quotient of the latter is abelian. Goursat therefore separates the CM and non-CM contributions. Repeated copies are acted on diagonally by their isogeny-class group. For non-CM copies, symplectic invariant theory gives degree-two pairings. For CM copies, independent norm-one characters give exactly opposite-weight pairings, also in degree two. The rational degree-two invariant space is $\mathrm{NS}(A^n)_{\mathbf{Q}}$. Rational descent as in Lemma P.2 proves the conclusion. Graphs of homomorphisms between isogenous copies and the corresponding Poincare divisors provide its geometric generators. $\square$

Example 21.2. Let E be $y^2 = x^3 - x$; then $\mathrm{End}^0(E) = \mathbf{Q}(i)$ and take $A = E^6$. The diagonal norm-one torus has characters $\chi, -\chi$ on two six-dimensional multiplicity spaces. In degree $2p$, an invariant selects p vectors from each space, so

$$\dim_{\mathbf{Q}} B^p(A) = \binom{6}{p}^2, \quad P_A(t) = 1 + 36t + 225t^2 + 400t^3 + 225t^4 + 36t^5 + t^6.$$

The divisor space is the 36-dimensional space of Hermitian 6×6 matrices over $\mathbf{Q}(i)$, represented by factor polarizations and Poincare classes associated with the endomorphisms 1 and i between factors. Its products generate every Hodge class. Here $\mathrm{End}^0(A) = M_6(\mathbf{Q}(i))$; this differs from the independent elliptic example but belongs to the same numerical partition. The distinction illustrates why a dimension partition alone cannot fix Hodge-space dimensions.

Conditional conclusion and external dependencies

Theorem P.23 (Conditional sixfold conclusion). Assume the split-eightfold and CM assertions stated in Theorem P.9, the specified full Kuga–Satake algebraicity assertion used in Theorem 14.11, and the auxiliary-spin assertion (Aux) stated before Theorem 13.19, together with the classical external results cited in the corresponding rows. Then every complex abelian sixfold satisfies the rational Hodge conjecture.

Proof. Poincare complete reducibility and the endomorphism classification reduce, up to isogeny, to the twenty-one rows S1–S11 and N1–N10. The split-eightfold input completes S7, S8 and S10 through Theorem P.11; the other simple rows and N1, N4–N10 are treated in their sections. Corollaries 13.1 and 14.1 handle the remaining CM-fourfold strata. The non-CM residual models of N2 and N3 are covered by Theorems 13.19 and 14.11, respectively. The independent and other nonresidual strata of those rows were already proved. Rational algebraicity is invariant under isogeny, so these arguments cover every sixfold. $\square$

For clarity, the new external dependencies are as follows.

External assertion	Application in this manuscript
Split-eightfold Weil algebraicity [14]	S7, S8, S10, via stabilization and contraction
CM abelian Hodge theorem [15]	CM-fourfold strata; CM transfer in residual N2
Specified complete KS correspondence [17]	Residual N3, via algebraic matrix coefficients
Auxiliary degree-four spin tensor [19]	Residual N2, via determinant stabilization

These inputs have different logical roles. In particular the CM theorem does not apply directly to a product with a non-CM fourfold. Theorem 13.19 first obtains a cycle on $B \times D$ from (Aux), and only then invokes the CM theorem on $D \times P$. Theorem 14.11 uses the specified KS map through algebraic weight-one factor maps; no abstract K3 Hodge isometry is declared algebraic. The auxiliary-spin input is not deduced here from the ordinary KS input.

All twenty-one sixfold rows are now covered relative to the stated inputs. The all-powers conclusions are those explicitly proved in the individual sections, including the residual N2 and N3 models; the sixfold conclusion alone does not assert an unrestricted all-powers theorem. The new external preprints and this manuscript still require independent mathematical review. In particular, the conditional conclusion is not an independent proof of the external assertions on which it depends.

References

- [1] A. P. Borselli, *Galois Groups of CM Fields in Degrees 24, 28, and 30*, Ph.D. thesis, Lehigh University, 2017, §2.2.2. Lehigh Preserve: official thesis record.
- [2] B. Moonen, *Notes on Mumford–Tate groups*, Centre Emile Borel, Paris, March 1999, especially §§2–3. <https://www.math.ru.nl/~bmoonen/Lecturenotes/CEBnotesMT.pdf>.
- [3] B. Moonen and Yu. G. Zarhin, *Hodge classes on abelian varieties of low dimension*, *Math. Ann.* **315** (1999), 711–733. <https://arxiv.org/abs/math/9901113>.
- [4] J. S. Milne, *Introduction to Shimura Varieties*, revised notes, Theorem 3.3(b) and the PEL moduli interpretation. <https://www.jmilne.org/math/xnotes/svi.pdf>.
- [5] B. Moonen, *Special subvarieties arising from families of cyclic covers of the projective line*, *Doc. Math.* **15** (2010), 793–819, Lemma 2.7, Table 1 and Proposition 4.7. <https://arxiv.org/abs/1006.3400>.
- [6] J. J. Ramón Marí, *On the Hodge conjecture for products of certain surfaces*, *Collect. Math.* **59** (2008), 1–26. The product-of-abelian-surfaces result is Proposition 2.18 in the 2005 preprint (Theorem 3.15 in the published numbering). <https://arxiv.org/abs/math/0505357>.
- [7] C. Vial, *Generic cycles, Lefschetz representations, and the generalized Hodge and Bloch conjectures for abelian varieties*, especially §3.2. <https://arxiv.org/abs/1803.00857>. <https://www.math.uni-bielefeld.de/~vial/SymAbelian.pdf>.
- [8] J. S. Milne, *Lefschetz classes on abelian varieties*, *Duke Math. J.* **96** (1999), 639–675, §5; see also the author’s erratum concerning Proposition 5.9. <https://www.jmilne.org/math/articles/1999a.html>.
- [9] J. S. Milne, *The Tate and standard conjectures for certain abelian varieties*, arXiv:2112.12815, Corollary 1.11. <https://www.jmilne.org/math/articles/tcav2.pdf>.
- [10] C. Yu and Z. Zheng, *Commensurability among Deligne–Mostow monodromy groups*, arXiv:2411.09682v2 (2025), Proposition 7.4. <https://arxiv.org/abs/2411.09682v2>.
- [11] E. Markman, *Cycles on abelian $2n$ -folds of Weil type from secant sheaves on abelian n -folds*, arXiv:2502.03415v2 (2025), Theorem 1.5.1 and Corollary 1.6.1. <https://arxiv.org/abs/2502.03415v2>.
- [12] H. Yanai, *On the index of degeneracy of a CM abelian variety*, *J. Théor. Nombres Bordeaux* **27** (2015), 815–820, §1. <https://www.numdam.org/articles/10.5802/jtnb.925/>.
- [13] E. Markman, *Secant sheaves and Weil classes on abelian varieties*, arXiv:2509.23403v2 (2026), Theorems 1.2, 1.4 and Corollary 1.3. <https://arxiv.org/abs/2509.23403v2>.
- [14] OpenAI, *Algebraicity of Weil classes on split abelian eightfolds*, OpenAI Math Release preprint, September 18, 2026, Theorem 1.1; public release October 6, 2026. Official manuscript and release record.
- [15] OpenAI, *The rational Hodge conjecture for CM abelian varieties*, OpenAI Math Release preprint, September 30, 2026, Theorem 1.1; public release October 6, 2026. Official manuscript and release record.
- [16] OpenAI, *Weil classes and Hodge classes on abelian powers*, OpenAI Math Release preprint, September 30, 2026, Lemma 3.2; public release October 6, 2026. Official manuscript and release record.
- [17] OpenAI, *Algebraicity of Kuga–Satake Correspondences for K3 Surfaces*, OpenAI Math Release preprint, October 3, 2026, Theorem 1.1; public release October 6, 2026. Used as the specified complete even-Clifford correspondence input in Theorem 14.11. Official manuscript and release record.
- [18] D. Huybrechts, *Lectures on K3 Surfaces*, Cambridge University Press, 2016; author lecture-note version, Chapters 4, 6 and 14. <https://www.math.uni-bonn.de/people/huybrech/K3Global.pdf>.
- [19] OpenAI, *The rational Hodge conjecture for products of K3 surfaces*, preprint, October 4, 2026, section *Degree-four tensors in spin families*, auxiliary-spin proposition. Source pinned at commit `adc7f1241b42e322a6451854ab7e4b4c146bf78a`; the exact assertion used is (Aux) preceding Theorem 13.19. Pinned source of the auxiliary-spin proposition.