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Conditional Review of the Twenty-One Cases

The rational Hodge conjecture for abelian sixfolds

Manuscript: `sixfold_hodge_revised`, research draft dated October 8, 2026.
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Central finding: assuming all OpenAI external assertions explicitly stated in the manuscript are correct, this review found no major mathematical gap that would invalidate the main conclusions in the twenty-one cases.

The key steps of the main proofs currently fit together. No gap was found that would require an additional unknown algebraicity theorem or replacement of a principal construction. This is not formal certification of every formula or auxiliary script, but neither is it acceptance based merely on the manuscript’s final “Proved” labels.

1. Review assumptions: four external assertions

The review assumes the following assertions hold with the strength stated in the manuscript:

1. Algebraicity of split eightfold Weil classes, as stated in P.9.
2. The rational Hodge conjecture for CM abelian varieties in arbitrary dimension.
3. Algebraicity of the specified complete even-Clifford Kuga–Satake correspondence, stated on manuscript page 51.
4. Algebraicity of the Aux degree-four spin tensor, stated on manuscript pages 44–45.

The last assertion is a separate input; general algebraicity of Kuga–Satake correspondences alone cannot replace it. The manuscript explicitly distinguishes these inputs. The review examines the author’s subsequent deductions under these assumptions, rather than rechecking the external assertions themselves.

2. The five formerly most difficult cases

Case	Decisive step	Review finding
S7	Start with a Weil plane for an imaginary quadratic subfield; use quaternionic endomorphisms and sum maps to generate the seven-dimensional exceptional space and volume invariants on powers	No major gap found
S8	Determine the full or special unitary group; use P.11 in the balanced case for every discriminant	No major gap found
S10	P.22 determines the derived group and determinant torus; P.11 handles the exceptional Weil plane and 9.3 passes to powers	No major gap found
N2	Aux gives a nonzero mixed cycle involving the fourfold and an auxiliary CM factor; a CM correspondence transfers it to the prescribed surface	No major gap found ; Aux is essential
N3	Construct a K3-type structure, extract matrix components of the specified KS correspondence in the complete Clifford module, and contract to a nonzero mixed cycle	No major gap found ; requires the specified complete KS correspondence

S7, S8 and S10 share P.11. N2 and N3 construct their mixed cycles separately; their arguments cannot simply be replaced by the conclusions for the first three cases. The principal checks follow.

2.1 P.11: recovering an arbitrary sixfold Weil plane from a split eightfold

Let A be a sixfold of Weil type with Hermitian determinant $d < 0$. Add two CM elliptic curves of opposite k -types and give their product P the Hermitian form

$$h_P = \text{diag}(1, d).$$

Then

$$X = A \times P, \quad \text{sig}(h_X) = (4, 4), \quad \det(h_X) = d^2 \in N_{k/\mathbb{Q}}(k^\times).$$

P.10 establishes hyperbolicity, so the hypotheses of the external eightfold theorem are met.

After obtaining algebraicity of $W_k(X)$, the manuscript uses an actual algebraic correspondence:

$$C_\beta(\gamma) = (\text{pr}_A)_*(\gamma \cup \text{pr}_P^* \beta), \quad 0 \neq \beta \in W_k(P) \subset \text{NS}(P)_{\mathbb{Q}}.$$

The contractions on both conjugate components are nonzero. Hence

$$C_\beta : W_k(X) \xrightarrow{\sim} W_k(A).$$

The cycle codimension is correctly $4 + 1 - \dim P = 3$. **There is no missing algebraic correspondence hidden behind an abstract Hodge isomorphism at this step.** The quaternionic generation argument in S7 and the determinant-invariant arguments in S8 and S10 can therefore proceed.

2.2 N2: the Aux hypotheses, nonvanishing and CM transfer

The auxiliary quadratic space in Theorem 13.19 satisfies

$$F = K^+ \text{ is real quadratic,} \quad \dim_F V' = 13 \equiv 5 \pmod{8},$$

and $\text{sig}(V'_\sigma) = (2, 11)$ at every real embedding σ . Local isotropy and Hasse–Minkowski split a hyperbolic plane. The complement still has real signature $(1, 10)$, so a second hyperbolic plane splits. Thus the Witt-index requirement in Aux is genuinely verified, not merely its dimension requirement.

The reviewer independently recomputed the exterior-algebra operation in N2.4:

$$C_U(e_{13} \otimes e_{23}) = e_3 \otimes e_{123} - e_{123} \otimes e_3.$$

Antisymmetrizing the input still gives a nonzero output. Consequently, the required nonvanishing

$$0 \neq z \in T_B \otimes H^2(D, \mathbb{Q})$$

is not an unchecked assertion. The manuscript also explains why Clifford operators in the other directions cannot cancel this projection, and how nonvanishing for complex selectors implies the existence of rational selectors.

The CM theorem is applied to $D \times P$, not incorrectly to $B \times P$, which contains a non-CM factor. It realizes the selected Hodge map

$$\varphi : H^2(D, \mathbb{Q}) \longrightarrow H^2(P, \mathbb{Q})$$

by an algebraic correspondence and transfers the previously obtained cycle to $B \times P$. Since φ is an isomorphism on the relevant simple substructure, $(1 \otimes \varphi)z \neq 0$. No problem was found with either nonvanishing or the codimension calculation in this transfer.

2.3 N3: extracting complete KS components and a nonzero contraction

The constructed K3-type structure satisfies

$$\dim_{\mathbb{Q}} W = 8, \quad h(W) = (1, 6, 1), \quad \text{sig}(q) = (2, 6).$$

The absence of rational $(1, 1)$ classes prevents an additional Picard-number increase when W is realized as K3 transcendental cohomology. The lattice inequality

$$\ell(A_{\Lambda_W}) + 2 \leq 10 < 14 = 22 - 8$$

meets the cited Nikulin criterion. No omitted lattice-embedding condition was found.

Next, $\det L = u^{-2}$ and $t = u^{-1}$ provide the Spin lift of the actual joint Hodge group. The complete Clifford regular module is used, and rational abelian factors are obtained through scalar extension of rational Hom spaces. The argument does not simply treat an individual complex half-spin representation as a rational abelian factor of the same dimension.

The reviewer checked the contraction in N3.4–N3.6:

$$\begin{aligned}\alpha_1(e_1) &= b_1 u^{-1}, & \alpha_1(e_2) &= b_2 u^{-1}, \\ \alpha_2(f_1) &= b_2 v^{-1}, & \alpha_2(f_2) &= -b_1 v^{-1}.\end{aligned}$$

Contracting against q^{-1} and pulling back along the diagonal of B yields a nonzero multiple of

$$(b_1 \wedge b_2) \otimes u^{-1} \otimes v^{-1}.$$

Since $\det V_1 = uv$, this lies in the target mixed invariant space. The algebraic transcendental projector on the K3 surface implements the contraction, and bilinearity gives descent to a rational cycle. Each of these steps is supported by an argument. **N3 does not currently lack the crucial step converting the KS correspondence into the required target cycle.**

3. The original sixteen cases

For S1–S6, S9, S11, N1 and N4–N10, this review likewise found no major mathematical gap in the main proofs.

S1–S6. The main checks concern completeness of the low-dimensional representation candidates, correct exclusion of diagonal couplings, and descent from quadratic invariant generation to rational divisor classes. No apparent mismatch was found between P.2–P.4 and their applications.

S9 and S11. The polarization adjustments do more than alter real signatures. P.17 calculates the resulting rational norm class; P.18 explicitly constructs a three-dimensional totally isotropic k -subspace. The splitting hypothesis required by Markman’s split-sixfold theorem is therefore actually proved.

The S11 rank bound. The reviewer checked Lemma 11.6’s exclusion of a four-dimensional orbit space and independently recomputed the three finite models in Remark 11.8. Their stabilizer orders are 96, 32, 96, and their orbit-span dimensions are 3, 4, 1, respectively, agreeing with the manuscript. This supports exclusion of rank four. Algebraicity of the Weil classes still comes from P.18, not from the rank computation alone.

N4, the 3 + 3 case. Lemma 15.2 does not merely compare abstract isogeny classes of tori: it tests whether the actual Hodge cocharacters satisfy a proposed common-quotient relation, in particular excluding the potential two-dimensional CM coupling. The remaining one-dimensional determinant relation produces precisely the stated mixed Weil plane, handled by odd-rank polarization adjustment.

N1, N5 and N6. Mixed classes are treated by odd-rank adjustment or pullbacks of fourfold Weil classes. **N7–N9** use the theorem for products of low-dimensional factors; **N10** uses quadratic invariant generation for elliptic products. The argument does not infer the Hodge conjecture for arbitrary products merely from the conjecture for each individual factor.

4. Two necessary boundaries

First: the sixfold itself versus all self-powers. The global theorem concerns the rational Hodge conjecture for every abelian sixfold itself. Assertions for powers must remain restricted to the scope explicitly proved in the individual sections. The global theorem does not additionally establish the conjecture for all A^n for every sixfold A . The manuscript’s conclusion already makes this distinction.

Second: no major gap found versus certification of all details. The finding is “**no major gap found**”, not “all details certified.” The reviewer did not reprove the external theorems or run every verification script mentioned in the manuscript.

Final assessment: under the specified external assumptions, this version can be regarded as a complete conditional proof draft covering the twenty-one cases. This review found no remaining major mathematical gap in the author's main proofs. In particular, withdrawal of the external results alone is not a reason to classify S7, S8, S10, N2 or N3 as lacking an internal conditional proof.

Source pointers and provenance

- Reviewed manuscript: the user-supplied *Hodge classes on abelian sixfolds: constructions and a conditional proof* (sixfold_hodge_revised). Proposition numbers and page references follow the supplied review.
- K3 lattices and periods: Daniel Huybrechts, public K3 lecture notes.
- Markman's cited results: arXiv:2502.03415.
- Products of low-dimensional factors: Ramón Marí, arXiv:math/0505357.

Editorial note: this is an English translation of the user-supplied external review. Repeated file-reference placeholders have been removed and identifiable links collected above. Red highlighting marks the central findings together with their external-assumption and review-scope qualifications.